PROJECTIVE PURE GEOMETRY

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PREFACE

An acquaintance with the principles and methods of Projective Pure Geometry is well-nigh indispensable to one who wishes to extend his mathematical studies into modern fields. For the more elementary student, an introduction to projective methods and some knowledge of the results obtained by their easier applications will add an interest to geometric study which is not readily developed by other means. In preparing the present text it has been my purpose to provide an approach to the study of projective geometry which will be found inviting and to present so much of the geometric material which may be most readily treated by projective methods as will stimulate the student to pursue his studies beyond the limits usually prescribed for undergraduates.

The definition adopted for the projective relation between two forms is that of Von Staudt, this being selected as the most elegant and, moreover, as placing most emphasis on purely geometric concepts. Free use has been made of Reye's *Geometrie der Lage*, the first part of which was translated into English by the present writer, with the author's approval, some thirty years ago. The order of presentation is largely that of Reye, but the subject matter has been entirely rewritten and, it is hoped, has been made more usable for textbook purposes. Many illustrative examples have been added and such theorems on conics and their properties have been selected as furnish a working knowledge of these curves and give an indication of the wide reach of the methods of pure geometry. A few space figures, as cones and ruled surfaces of the second order, have come naturally into consideration, but besides these, not many three-dimensional forms have been included

although space conceptions have been emphasized throughout.

It has not seemed desirable to include in the present text a discussion of the logical foundations on which modern geometry rests, but rather to assume the properties commonly assigned to geometric elements and proceed with the superstructure in the most direct way. The choice lay between this procedure and an extended presentation of foundations for which the student for whom the text is designed is seldom ready.

Much of the difficulty experienced in the study of projective geometry lies in the terminology employed and in the use of unfamiliar forms of statement. An effort has been made to relieve these hindrances by avoiding technical phraseology wherever colloquial language would serve the purpose. The terminology employed, for the most part, is in accord with standard English usage.

Projective Pure Geometry, or Modern Synthetic Geometry as it is frequently called, is a development of the past one hundred years. The foundation theorems and concepts of this study, such as harmonic division and the notion of infinitely distant elements, go back to earlier times; in fact, they antedate the beginnings of analytic geometry and some of them are to be found in the works of Apollonius and Pappus. But, following a lapse of a century and a half in which the new analytic methods held sway, it was not till the early years of the nineteenth century that interest in pure geometry began again to command the attention of creative minds.

Beginning with Monge (1746–1818) and Carnot (1753–1823), the outstanding names in this modern revival are Poncelet (1788–1867), Chasles (1793–1880), Steiner (1796–1863), and Von Staudt (1798–1867), whose publications

take rank as classics in pure geometry. To these may be added two others whose interpretations of the writings of the great masters have opened the way for more recent students to the beauties of geometric theory. I refer to Cremona (1830–1903) whose *Elementi di geometria projettiva* was first published in 1873 and translated into English in 1885, and Reye (1838–1919), of whom earlier mention has been made. It is to the writings of these two that the greatest indebtedness is felt in the preparation of this text.

For an early interest in the subject and for suggestions in the manuscript I am indebted to Professor Henry S. White of Vassar College, for many years my colleague in Northwestern University.

THOMAS F. HOLGATE

EVANSTON, ILLINOIS
January, 1930

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PROJECTIVE PURE GEOMETRY

CHAPTER I

PRIMITIVE FORMS—PROJECTION AND SECTION—INFINITELY DISTANT ELEMENTS

1. Introduction. Pure geometry is characterized by the fact that in its study the geometric concept is kept always in mind, while in analytic or coördinate geometry the geometric notions with which we start are expressed in algebraic language, then certain algebraic operations are performed, and the results are interpreted as new geometric relations.

Analytic geometry makes use of measurement or magnitude at every step. It is wholly metric, while pure geometry may be metric, as in the elementary geometry of Euclid, or it may be non-metric, as in the geometry of position of Von Staudt.[1]

For the most part, the present text will follow the method of Von Staudt as interpreted by Reye,[2] and will introduce metric relations only as it may seem desirable for the purpose of identifying the more general results found in the geometry of position with well-known particular properties developed by other methods.

2. Elements. The ***point,*** the ***straight line,*** and the ***plane*** are accepted as simple undefined elements of pure geometry. Each of them may be given consideration quite independ-

[1] Karl Georg Christian von Staudt (1798–1867), for many years professor in the University of Erlangen, is generally acknowledged to be the founder of pure geometry on a basis free from metric considerations. His monumental work, *Geometrie der Lage*, was published in 1847, at Nürnberg.

[2] Theodor Reye (1838–1919), professor at the University of Strassburg, extended and clarified the work of Von Staudt in his *Geometrie der Lage*, first published in 1866, at Hannover.

ently of the others and their properties are the same as are commonly assigned to them in elementary geometry.

Throughout the text we shall designate points by capital letters, A, B, C, $\cdots$; lines by small italics, a, b, c, $\cdots$; and planes by Greek letters, α, β, γ, $\cdots$. Whenever the term *line* is used without further description, it will mean *straight line*. The term *ray* will have the same significance.

The straight line and the plane are assumed to be unlimited in extent.

3. Primitive Forms. While each of the elements may be thought of apart from the others, each is the *base* or *support* of an indefinite or unlimited number of the others. An unlimited number of lines or points lie on a given plane, the plane being the support of the points or lines; an unlimited number of planes pass through a given line and an unlimited number of points lie on a given line, the line being the base or support of the planes or points; any given point is likewise the support of an unlimited number of lines or planes passing through it. The elements may thus be combined into certain *primitive forms* which occupy an important place in the modern study of geometry.

DEFINITIONS. (a) The totality of points on a line is called a *range of points* or a *point-row*. The individual points are the *elements of the range* and the relative positions of the elements are unchanged when the straight line or base is moved from one position to another.

(b) The totality of planes passing through a line is called a *pencil of planes* or an *axial pencil*. The line is the *axis* of the pencil. This form is sometimes called a *sheaf of planes*.

(c) The totality of straight lines or rays passing through a point and lying in one plane (Fig. 1) is called a *pencil of rays* or a *flat pencil*; it is also sometimes called a *sheaf of rays*. The point is the center of the pencil.

(*d*) The totality of rays in space through a point is called a *bundle of rays*. The point is the *center* of the bundle.

(*e*) The totality of planes through a point is called a *bundle of planes*. The point is the center of the bundle.

(*f*) The totality of points in a plane is called a *field of points*.

(*g*) The totality of lines or rays in a plane is called a *field of rays*.

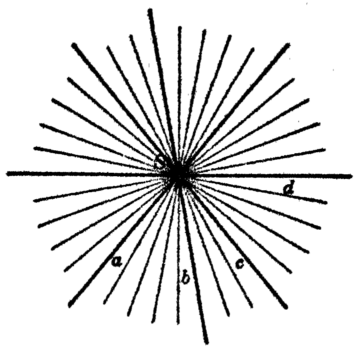


FIG. 1.

(*h*) The *points of space*, *planes of space*, and *lines of space* are primitive forms constituted, as their names imply, of the totality of points, lines, or planes, of unlimited space.

There are thus ten primitive forms consisting of the elements in various aggregates.

4. Projection and Section. When one looks at an object —a building, a picture, a geometric figure—every visible point of it sends a ray of light into the eye, and from the aggregate of these rays the observer obtains an image of the object. If a plane should be interposed between the eye and the object, each of these rays would mark a point on this plane, and from the aggregate of these points the observer would receive the same image as from the object itself. By such a process the object is said to be *projected* from the eye on the intervening plane, and the figure in the plane resulting from the aggregate of points is a *projection* of the object. Each ray is a *projecting ray* for some point of the object, and its intersection with the plane is the *trace* on the plane of the observed point. The aggregate of projecting rays may be called the *projector* of the object and

of this projector the intercepting plane makes a *section.* This method of studying an object is called the method of *projection and section* and of this method large use will be made in the following pages.

Since the projecting rays in this process converge to a point, the eye, the method is spoken of as *central projection* and the point is the *center of projection.* Parallel projection and orthogonal projection, which are much used in descriptive geometry and engineering, are special cases of central projection.

5. The Field of Projective Geometry. The process of central projection and section yields, on the plane of section, a trace or projection of the object under consideration in which shapes and magnitudes may be radically changed but in which certain properties are preserved. For example, those points of the object which lie on a straight line will give rise in the projector to rays lying in one plane, and the trace of these on the plane of section will lie again on a straight line; namely, on the line of intersection of the two planes. The projection of the points of a curved line will ordinarily be a curved line in the section, though the form and dimensions of the curve may be wholly changed. If a circle is projected from any point not in its plane, the projector is a cone with the point as vertex, and the section by any plane is some sort of conic section. If in the original figure a straight line cuts the circle in two points, or is tangent to the circle, the same relation between the straight line and the conic will appear in the projection. It is with properties such as these, which are unaltered by projection and section, that projective geometry is chiefly concerned.

6. Derivation of Primitive Forms from Each Other by Projection and Section. If a range of points is projected from any point outside itself, the projector is a pencil of

rays (Fig. 2), and any section of a pencil of rays by a straight line is a range of points.

A pencil of rays is projected from any point not in its plane by a pencil of planes whose axis is the ray joining the point and the center of the given flat pencil; any plane section of a pencil of planes is a pencil of rays.

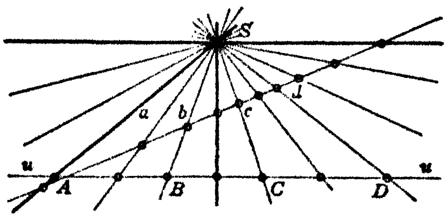


FIG. 2.

A plane section of a bundle of rays is a field of points; and a projection of a field of points is a bundle of rays.

A plane section of a bundle of planes is a field of rays; and a projection of a field of rays is a bundle of planes.

7. Infinitely Distant Elements. The assumption that straight lines are unlimited in extent makes necessary some sort of understanding regarding their inaccessible points.

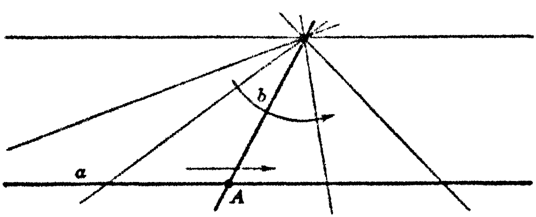


FIG. 3.

Suppose there are given, in a plane, two straight lines a and b which intersect at a point A (Fig. 3). If one of these lines a remains fixed, while the other b rotates steadily in a given sense about a fixed point different from A, the

point of intersection will move steadily along the line a in a given sense until it is lost to view. If the point A moves along the line to the right as indicated in the figure, and the rotation of b continues, the point of intersection of a and b disappears far to the right and presently re-appears far to the left, still moving to the right along a. Its passage from the extreme right to the extreme left, or *vice versa*, is subject to an assumption; namely, that for only one position of the line b is the intersection of a and b inaccessible.

For this position of the line b the point of intersection of a and b has become purely ideal. It is a postulated point which makes the line a such that it may be traversed from any one point of it to any other point of it in either sense without leaving the line. This point is spoken of as the ***ideal or infinitely distant*** point of the line, or as the ***point at infinity*** on the line, and it is treated in all our discussions like any other point of the line.[1] This conception is expressed in the following postulate which may be regarded as the fundamental assumption of projective geometry.

POSTULATE. *On any straight line there is one and only one ideal or infinitely distant point. This point makes the line such that it may be traversed either way along the line, from any one point of it to any other point of it without leaving the line.*

8. Parallel Lines. On the basis of the above assumption we are permitted to make a general statement regarding the intersections of lines in a plane which without such assumption would require that exceptional cases be recognized. This statement is that any two straight lines in a plane intersect either in an *actual* point or in an *ideal* or *infinitely distant* point.

[1] The notion of infinitely distant points and their introduction into geometry is attributed to Girard Desargues (1593–1662).

DEFINITION. Two straight lines which intersect in an infinitely distant point are said to be ***parallel***.[1]

Since on any line there is one and only one ideal or infinitely distant point and since through two points only one straight line can be drawn, it follows that through any given point not lying on it there can be drawn one and only one line parallel to a given line.

A system of parallel lines in a plane all pass through the same ideal point; such a system is therefore a pencil of rays whose center is infinitely distant.

The infinitely distant point of a line is said to determine its ***direction***. Parallel lines therefore have the same direction and two lines intersecting in a finite point have different directions since their infinitely distant or ideal points are different.

9. Points on a Line Separated by Other Points. Since a straight line is continuous through the infinitely distant point, any two points of it, A and B, divide the line into two segments, one of which contains the infinitely distant point. That segment of the line on which the infinitely distant point lies is called the ***major segment;*** the other, the ***minor segment.***

[1] Euclid's definition of parallel lines is purely negative; namely, they are lines lying in the same plane which produced ever so far either way do not meet. In his geometry the plane was limited in extent and a statement that two lines in a plane always intersect must be qualified by the provision that they are not parallel. To make progress in the theory of parallel lines there was need for some positive quality to be affirmed and hence his famous twelfth axiom:

If a straight line meets two straight lines in the same plane, so as to make the two interior angles on one side of it together less than two right angles, these straight lines if produced will meet on that side on which are the angles which are less than two right angles.

Having demonstrated from earlier definitions and assumptions that any two angles of a triangle are together less than two right angles (Bk.I, Prop. 17), in other words, that if a straight line meets two intersecting lines, the interior angles on one side are together less than two right angles, Euclid assumed the truth of the converse theorem and stated it as an axiom.

Definition. Two points, A and B, of a straight line are said to be separated by other points on the line when it is not possible to pass along the line from A to B, traversing either segment, without crossing one or more of those points.

A single point chosen in one segment of the line, therefore, will not separate A and B. It requires two points, one in each segment, to separate them.

If four points, A, B, C, D, are chosen in any manner on a straight line, each of these points is separated from one other, and from only one other, by the remaining two. Thus, in Fig. 4, the point A is separated from C by B and D, but is not separated from either B or D by the remaining two; so also B is separated from D by A and C, but not from A or C by the remaining two.

Fig. 4.

10. Infinitely Distant Points of a Plane Lie on a Straight Line. If a straight line rotates in a plane about a fixed point, for each position of the line there is an infinitely distant point on it. The locus of such points has the property that one and only one point of it lies on any actual line of the plane. The locus, therefore, may consistently be considered a straight line, since any path which can have only one point in common with a straight line is itself a straight line.

Definition. The infinitely distant points of a plane lie on a straight line, the so-called ***infinitely distant line*** of the plane.

A line which has on it more than one infinitely distant point can have no actual points, since any line through an actual point has only one infinitely distant point. Such a

line, therefore, lies wholly at infinity, and is the infinitely distant line of any plane through it.

That the infinitely distant line of a plane is purely ideal will be emphasized by the consideration that under the definition of parallel lines this line is parallel to every line of the plane since any line of the plane intersects it in an infinitely distant point.

Two intersecting planes have a straight line in common and if this line is infinitely distant the two planes are parallel.

11. Infinitely Distant Points of Space Lie on a Plane. Considerations similar to those of § 10 will fix all the infinitely distant points of space on a plane, the so-called infinitely distant or ideal plane. For, whatever may be the nature of the locus of these points, it has one and only one point in common with any actual ray of space and it is intersected by an actual plane in a straight line. The locus therefore possesses the essential properties of a plane.

Any system of parallel planes is a pencil of planes whose axis is an infinitely distant line, and to such a pencil the infinitely distant plane itself belongs.

EXERCISES

1. If two intersecting lines are projected from any point not in their plane, the projector consists of two intersecting planes. How can a section of those planes be taken to consist of two parallel lines? In other words, how can two intersecting lines be projected into two parallel lines?

2. Show that any quadrangle may be projected into a parallelogram.

3. Given two lines not lying in the same plane and hence not intersecting, through a given point construct a plane which is parallel to both of them.

4. If the faces of a tetrahedron, or those faces extended, are cut by a plane not passing through a vertex, the resulting figure consists of four

lines and the six points in which they intersect, two and two. How can the plane of section be chosen so that the figure of section will be a parallelogram?

5. If the vertices and edges of a tetrahedron are projected from a point not lying on any of its faces, the projector consists of four rays and six planes through the center of projection, two rays in each plane and three planes through each ray. A section of this projector consists of four points and the six lines joining them, two and two.

6. A circle is projected from a point outside its plane by a cone, any plane section of which, not passing through the vertex, will contain two infinitely distant points, or one, or none, according as the plane of section is parallel to two rays of the cone, or to one, or to none of those rays.

7. If a triangle lying in a given plane is projected on a second plane from a fixed point, each side of the triangle will intersect its projection on the second plane, and the three points of intersection are collinear.

8. If a line is parallel to a plane, their common point is infinitely distant. Any plane through one of two parallel lines is parallel to the other.

9. If a line is parallel to each of two intersecting planes, it is parallel to their line of intersection.

10. Through two given lines in space which do not intersect, draw a pair of parallel planes.

CHAPTER II

THE PRINCIPLE OF DUALITY—SIMPLE AND COMPLETE RECTILINEAR FIGURES

12. Reciprocity in Geometric Forms. Among the properties assigned to the elements in pure geometry, a certain reciprocity or duality may be observed which will have an important bearing on many subsequent developments.

For example, if we confine our attention to the elements in a single plane, the following statements are true and when placed side by side they call attention to an interesting relation.

(a) Two points determine a line.

(b) Two lines determine a point.

If the words *point* and *line* in either of these statements are interchanged, the other statement is the result. There is, therefore, a certain ***reciprocal*** relation between the statements, and the elements *point* and *line* appearing in them may be considered reciprocal or dual elements.

In the following parallel columns, examples are given of reciprocal geometric forms in a plane, the forms (a) and (b) being reciprocals.

1(a). Two points on a line.	1(b). Two lines through a point.
2(a). The totality of points on a line—a range of points.	2(b). The totality of lines through a point—a pencil of rays.
3(a). Three points not on the same line and the lines through them, two and two.	3(b). Three lines not through the same point and their points of intersection, two and two.

4(*a*). Four points, no three of which lie on the same line.

4(*b*). Four lines, no three of which pass through the same point.

5(*a*). Four points and the six lines determined by them, two and two.

5(*b*). Four lines and the six points determined by them, two and two.

6(*a*). A sequence of points of which not more than two lie on any line.

6(*b*). A sequence of lines of which not more than two pass through any point.

13. Duality in a Plane. Whenever a statement is made or a theorem is announced on the relative positions of points and lines in a plane, a reciprocal or dual statement or theorem may be obtained by interchanging the elements *point* and *line*, and if the original statement is true, the reciprocal statement is likewise true. This principle, which is known as the ***principle of duality***[1] and which plays a large part in the pure geometry of position, may be stated as follows.

PRINCIPLE OF DUALITY IN A PLANE. *From any theorem on the relative positions of points and lines in a plane, another theorem equally valid may be obtained by interchanging with suitable connectives the words point and line.*

The following theorems, of which the proofs will appear later, are illustrations of duality in a plane, the theorems (*a*) and (*b*) being reciprocals.

1(*a*). If two vertices of a variable triangle move on fixed lines while the sides pass always through three fixed collinear points, the third vertex

1(*b*). If two sides of a variable triangle pass through fixed points while the vertices lie always on three fixed concurrent lines, the third side

[1] The principle of duality was first asserted in definite form by Gergonne, *Annales de Mathématiques*, Vol. 16 (1826), though Poncelet had previously shown, *Traité des propriétés projectives des figures* (Paris, 1822), that to any figure in space a polar reciprocal figure may be constructed.

will likewise move on a fixed line.

2(*a*). If three points, A, C, E, are chosen at random on a fixed line p and likewise three points B, D, F, at random on a fixed line q, the intersections of the lines AB and DE, BC and EF, CD and FA, lie on one straight line.

will likewise pass through a fixed point.

2(*b*). If three rays, a, c, e, are drawn at random through a fixed point P and likewise three rays, b, d, f, at random through a fixed point Q, the lines joining the points (ab)[1] and (de), (bc) and (ef), (cd) and (fa), pass through one point.

14. Duality in Space. In forms not confined to a single plane the reciprocal relations among the elements take on a different aspect. For example, in three-dimensional geometry, the following statements are true.

1(*a*). Three points not lying on the same line determine a plane.

2(*a*). Two points determine a line; namely, the line passing through them.

3(*a*). On any line there lie an unlimited number of points.

4(*a*). A line and a plane not passing through it determine a point.

5(*a*). Two lines which have a common point lie in one plane.

1(*b*). Three planes not passing through the same line determine a point.

2(*b*). Two planes determine a line; namely, their line of intersection.

3(*b*). Through any line there pass an unlimited number of planes.

4(*b*). A line and a point not lying on it determine a plane.

5(*b*). Two lines which lie in one plane have a common point.

In these statements, the duality is between *point* and *plane*, on the one hand, and between *line determined by two points* and *line determined by two planes* on the other. In

[1] The symbol (ab) here denotes the point of intersection of the lines a and b.

any of the statements of the left-hand column, if the elements *point* and *plane* are interchanged, and also *line* and *line*, the corresponding statement of the right-hand column is the result.

In three-dimensional geometry, then, point and plane may be considéred dual or reciprocal elements; also, line determined by two points and line determined by two planes are reciprocal elements.

As applied to three-dimensional figures, the principle of duality may be stated as follows.

PRINCIPLE OF DUALITY IN SPACE. *From any theorem on the relative positions of points, lines, and planes in a geometric configuration in three dimensions, another theorem equally valid may be obtained by interchanging with suitable connectives, the words point and plane, line and line.*

The principle of duality either in a plane or in space is not, in general, applicable where metric relations are involved. It is only in respect to positional relations that it has significance.

15. Reciprocal Theorems in Space. As illustrations of the principle of duality in space, the following theorems will serve, reciprocal theorems being placed side by side.

1(*a*). If four points, A, B, C, D, are so situated that the lines AB and CD meet in a point, then the four points lie in one plane, and the lines AC and BD, also AD and BC, likewise meet in a point.

2(*a*). If any number of straight lines are so situated that each intersects every other, then the lines must all pass through one point or they

1(*b*). If four planes α, β, γ, δ, are so situated that the lines $(\alpha\beta)$ and $(\gamma\delta)$ lie in a plane, then the four planes meet in one point, and the lines $(\alpha\gamma)$ and $(\beta\delta)$, also $(\alpha\delta)$ and $(\beta\gamma)$, likewise lie in a plane.

2(*b*). If any number of straight lines are so situated that each lies in a plane with every other, then the lines must all lie in one plane or they

must all lie in one plane.

must all pass through one point.

3(*a*). Four points lying in one plane determine six lines; namely, the lines joining them, two and two.

3(*b*). Four planes passing through one point determine six lines; namely, the lines of intersection of the planes, two and two.

The solutions of the following problems are reciprocal.

1(*a*). To draw a straight line through two given points.

1(*b*). To find the line of intersection of two given planes.

2(*a*). To pass a plane through a given line and a point not lying on it.

2(*b*). To find the point of intersection of a line and a plane not passing through it.

3(*a*). Through a given point to draw a line intersecting two given lines which do not lie in the same plane.

SOLUTION. The given point with each of the given lines determines a plane. The line of intersection of these two planes will pass through the given point and will meet each of the given lines.

3(*b*). In a given plane to draw a line intersecting two given lines which do not pass through the same point.

SOLUTION. The given plane with each of the given lines determines a point. The line joining these two points will lie in the given plane and will meet each of the given lines.

16. Duality in the Primitive Forms. If attention is confined to the geometry of a single plane, there are only four of the ten primitive forms which come under consideration; namely, the range of points and the pencil of rays, the field of points and the field of rays. Of these the first two are reciprocals in the plane, as are also the last two.

In three-dimensional geometry the dual or reciprocal relation between primitive forms appears in the following parallel columns in which reciprocal forms are placed side by side and are numbered (*a*) and (*b*), respectively.

1(*a*). **Range of points**—the system of points on a given line.

2(*a*). Pencil of rays—the system of rays through a point lying in one plane.

3(*a*). Bundle of rays—the totality of rays through a point.

4(*a*). Bundle of planes—the totality of planes through a point.

5(*a*). The totality of points in space.

6(*a*). The totality of lines in space, each determined by two points.

1(*b*). **Pencil of planes**—the system of planes through a given line.

2(*b*). Pencil of rays—the system of rays in a plane passing through one point.

3(*b*). Field of rays—the totality of rays in a plane.

4(*b*). Field of points—the totality of points in a plane.

5(*b*). The totality of planes in space.

6(*b*). The totality of lines in space, each determined by two planes.

It will be observed that the pencil of rays and the totality of lines in space are self-reciprocal forms.

17. Duality in the Regular Solids of Elementary Geometry. A study of the reciprocal relations among the five so-called regular solids of elementary geometry is interesting.

Tetrahedron. A tetrahedron, or tetragon, consists of four points (vertices) not lying in one plane and the planes (faces) determined by them, three and three; also the straight lines (edges) joining the points, two and two.

The tetrahedron has four vertices, four faces, and six edges. The faces meet by threes in the vertices, and the

Tetrahedron. A tetrahedron consists of four planes (faces) not passing through one point and the points (vertices) determined by them, three and three; also the straight lines (edges) in which the planes intersect, two and two.

The tetrahedron has four faces, four vertices, and six edges. The vertices lie by threes in the faces, and the

vertices lie by threes in the faces; the edges lie by threes in the faces and pass by threes through the vertices.

faces meet by threes in the vertices; the edges pass by threes through the vertices and lie by threes in the faces.

The tetrahedron is a self-reciprocal figure, determined either by its four vertices or by its four faces.

CUBE. A cube has eight vertices, six faces, and twelve edges. The vertices lie by fours in the faces; the faces meet by threes in the vertices; the edges lie by fours in the faces and pass by threes through the vertices.

OCTAHEDRON. An octahedron has eight faces, six vertices, and twelve edges. The faces meet by fours in the vertices; the vertices lie by threes in the faces; the edges pass by fours through the vertices and lie by threes in the faces.

The cube and the octahedron are thus seen to be reciprocal or dual figures.

A similar study will show that the dodecahedron and the icosahedron are reciprocal figures.

18. Primitive Forms of Different Dimensions. The reciprocal relations existing among the geometric primitive forms and their derivation one from another by projection and section suggest a grouping of these forms into classes, the forms in each class having certain distinctive properties in common.

The three forms, range of points, pencil of rays, and pencil of planes, may be derived from one another by projection and section or by reciprocation, and for this reason they may properly be considered as belonging to the same general class. These three forms are designated as *primitive forms of the first order*, or as *one-dimensional primitive forms*, and each of them may be said to consist of a single infinity of elements.

The field of points, field of rays, bundle of rays, and bundle of planes are similarly related, being derivable, one from another, either by projection and section or by reciprocation. They are designated as *primitive forms of the second order*, or *two-dimensional*, and may be said to consist of a two-fold infinity of elements.

The points of space and planes of space are reciprocal forms of the *third order*, or *three-dimensional*, and consist of a three-fold infinity of elements, while the totality of lines of space as a primitive form is self-reciprocal and of the *fourth order*, consisting of a four-fold infinity of rays.

19. Rectilinear Figures. A plane rectilinear figure consists of a given number of straight lines lying in one plane and their points of intersection; or, reciprocally, of a given number of points in a plane and the lines determined by them. The given lines are the ***sides*** of the figure, and their intersections, two and two, are the ***vertices;*** or reciprocally, the given points are the ***vertices*** of the figure, and the lines joining them, two and two, are the ***sides.***

A simple rectilinear figure consists of a given number of lines and their intersections, two and two, *in succession;* or, reciprocally, of a given number of points and the lines joining them, two and two, *in succession.* The number of sides in a simple rectilinear figure is the same as the number of vertices, and the reciprocal of a simple figure is similarly composed.

A complete rectilinear figure consists of a given number of lines and *all* their intersections, two and two; or, reciprocally, of a given number of points and *all* the lines joining them, two and two.

Since a triangle consists of three lines and their intersections, two and two, or of three points and the lines joining them, two and two, it is self-reciprocal in the plane, and it is

at the same time both a simple and a complete rectilinear figure.

20. Simple and Complete Quadrangles and Quadrilaterals. The following figures are reciprocal to each other in a plane.

A ***complete quadrangle*** consists of four points, A, B, C, D, and the six lines joining them, two and two (Fig. 5). The four points are the ***vertices*** and the six lines are the ***sides*** of the complete quadrangle. The sides fall into three pairs of so-called ***opposite sides,*** the line joining two vertices and the line joining the other two forming a pair.

A ***complete quadrilateral*** consists of four lines, a, b, c, d, and their six points of intersection, two and two (Fig. 6). The four lines are the ***sides*** and the six points are the ***vertices*** of the complete quadrilateral. The vertices fall into three pairs of so-called ***opposite vertices,*** the intersection of two sides and the intersection of the other two forming a pair.

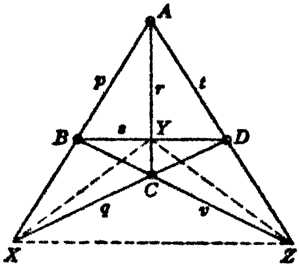


FIG. 5.

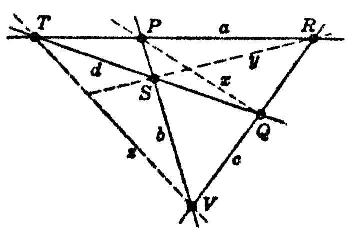


FIG. 6.

The pairs of opposite sides, by their intersections, determine three additional points X, Y, Z, called ***diagonal points.***

Thus a complete quadrangle has four vertices, six sides, and three diagonal points.

The pairs of opposite vertices determine three additional lines, x, y, z, called ***diagonals.***

Thus a complete quadrilateral has four sides, six vertices, and three diagonals.

In either case the triangle XYZ (xyz) is called the ***diagonal triangle*** of the figure.

A ***simple quadrangle*** consists of four vertices and the four lines joining them in succession; while a ***simple quadrilateral*** consists of four lines and the points of intersection of those lines in succession.

A simple quadrangle and a simple quadrilateral are consequently like figures; each has two pairs of opposite vertices and two pairs of opposite sides. The line joining a pair of opposite vertices is a ***diagonal*** and the point of intersection of a pair of opposite sides is a ***diagonal point.***

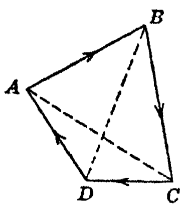


FIG. 7.

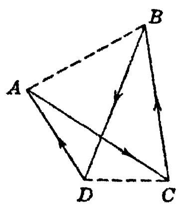


FIG. 8.

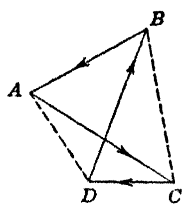


FIG. 9.

Three different simple quadrangles are determined by the same four points A, B, C, D, according as the points are taken successively in one order or in another. Thus, $ABCD$, $ACBD$, $ACDB$, are different simple quadrangles having the same vertices.

The simple quadrangle $ABCD$ has AB and CD, AD and BC, as pairs of opposite sides; A and C, B and D, as pairs of opposite vertices; AC and BD as diagonals. The quadrangle $ACBD$ has AC and BD, AD and CB, as pairs of opposite sides; A and B, C and D as opposite vertices; AB and CD as diagonals; while the quadrangle $ACDB$ has AC and DB, AB and CD, as pairs of opposite sides; A and D, B and C as opposite vertices; AD and CB as diagonals.

Similarly, the same four lines a, b, c, d, determine three different simple quadrilaterals according as they are taken successively in the order *abcd*, *acbd*, or *acdb*.

21. Simple and Complete Plane Figures in General.

A ***complete n-angle or n-gon*** consists of n points in a plane, no three of which lie on the same straight line, and the lines joining them, two and two.

Thus a complete n-gon has n vertices and $\frac{1}{2}n(n-1)$ sides. Through each vertex there pass $(n-1)$ sides.

A ***complete n-lateral or n-side*** consists of n lines in a plane, no three of which pass through the same point, and the points in which they intersect, two and two.

Thus a complete n-side has n sides and $\frac{1}{2}n(n-1)$ vertices. On each side there lie $(n-1)$ vertices.

A simple n-angle or n-gon consists of n points and the lines joining them, two and two, in succession, while a simple n-lateral or n-side consists of n lines and their points of intersection, two and two, in succession. Thus a single n-gon and a simple n-side are reciprocal figures, each consisting of n vertices and n sides.

In a simple n-gon the lines joining consecutive vertices are sides and the lines joining non-consecutive vertices are diagonals.

A simple n-gon has n vertices, n sides, and $\frac{1}{2}n(n-3)$ diagonals.

In any complete n-gon there are $\frac{1}{2}(n-1)!$ simple n-gons.

A simple n-gon with all its diagonal forms a complete n-gon.

In a simple n-side the points of intersection of consecutive sides are vertices and the points of intersection of non-consecutive sides are diagonal points.

A simple n-side has n sides, n vertices, and $\frac{1}{2}n(n-3)$ diagonal points.

In any complete n-side there are $\frac{1}{2}(n-1)!$ simple n-sides.

A simple n-side with all its diagonal points forms a complete n-side.

A simple n-gon or n-side has $2n$ elements, vertices and sides, of which any two elements are opposite when they are separated by the same number of elements enumerated either way around the figure. If n is even, a side is always opposite a side and a vertex is opposite a vertex; if n is odd, a side is opposite a vertex and a vertex is opposite a side. For example, in a simple pentagon each vertex is opposite a side, while in a simple hexagon each vertex is opposite a vertex and each side opposite a side.

22. Projections of Simple and Complete Plane Figures. A simple quadrangle projected from a point not in its plane gives a simple figure in a bundle consisting of four edges through the point and the four planes through those edges, two and two, in succession; and a complete quadrangle projected in the same way gives a four-edged figure in a bundle with the six planes through those edges, two and two.

If a simple n-gon (or n-side) lying in a plane is projected from any point not in that plane, we obtain a simple n-edge (or n-face) lying in a bundle. This consists, in either case, of n rays of the bundle and the planes determined by them, two and two, in succession. A complete n-gon (or n-side) projected from any point not in its plane will give a complete n-edge (or n-face) lying in a bundle, consisting of n rays (or planes) of the bundle and all the planes (or rays) determined by them, two and two.

EXERCISES

1. A figure consists of three points on each of two lines in a plane, and the lines joining them, two and two. How many such lines are there? What is the plane reciprocal of the figure?

2. What is the space reciprocal of a triangle?

3. If four planes pass through a given point and no three of them contain the same line, how many lines of intersection are there and how are they situated relative to each other? What is the space reciprocal of this figure? Make a diagram of a plane section of the figure.

4. Four planes are chosen at random, not all through the same point and no three through the same line. How many points and lines of intersection are there, and how do they lie relative to each other and to the chosen planes? What is the reciprocal of this figure?

5. In the regular dodecahedron of elementary geometry, how many vertices, edges, and faces are there? How many in the icosahedron?

6. Through any point how many lines may be drawn to intersect two skew lines, that is, lines which do not lie in the same plane? Show that of the lines intersecting three skew lines no two lie in the same plane.

7. Defining a plane polygon as regular when its consecutive sides are equal and make equal angles with each other, show that the vertices of a regular pentagon determine two different regular pentagons, according as the vertices are taken to be consecutive in one order or in another. Do the vertices of a regular hexagon or a regular heptagon determine more than one such figure? Illustrate the result by a drawing.

8. Assuming the correctness of the following theorem, state its reciprocal:

In a complete quadrangle, the sides of the diagonal triangle intersect the sides of the quadrangle in six points other than the vertices of the triangle, which lie by threes on four straight lines.

9. The simple hexagon $ABCDEF$ is such that the diagonals AD, BE, and CF pass through one point. What is the plane reciprocal figure?

10. A configuration consists of two triangles $A_1B_1C_1$ and $A_2B_2C_2$ so situated that the lines A_1A_2, B_1B_2, C_1C_2 pass through one point. Describe the space reciprocal of this configuration (1) when the given triangles lie in the same plane; (2) when they lie in different planes.

CHAPTER III

HARMONIC FORMS

23. Rectilinear Figures in Perspective. Two rectilinear figures of the same number of sides and vertices may be correlated to each other by relating each vertex of the one to a particular vertex of the other, by which means each side of the one is related to a particular side of the other. The vertices and sides so related in the two figures are said to ***correspond*** or to be ***homologous.*** For convenience of notation, corresponding vertices and sides in correlated figures will be marked by the same letters.

Correlated rectilinear figures are ***in perspective*** when they are so situated that the lines joining pairs of corresponding vertices are concurrent.

24. Desargues' Theorem on Perspective Triangles.[1] *If two correlated triangles are so situated that the lines joining pairs of corresponding vertices meet in a point, the pairs of corresponding sides will intersect in three points of one straight line.*

To prove Desargues' theorem let us take first the case in which the given triangles, $A_1B_1C_1$ and $A_2B_2C_2$ lie in different planes.

Since it is given that the lines A_1A_2 and B_1B_2, joining pairs of corresponding vertices, meet in a point (Fig. 10), say the point O, they determine a plane in which will lie the corresponding sides A_1B_1 and A_2B_2. These sides therefore intersect in some point X.

[1] This theorem is usually attributed to Desargues (1593–1662), though in fact it was announced by Euclid (Pappus, *Mathematicae Collectiones*, preface to Book VII).

Also, since A_1A_2 and C_1C_2 meet in the point O, they determine a plane in which lie the correlated sides A_1C_1 and A_2C_2, intersecting in some point Y.

Similarly, since B_1B_2 and C_1C_2 meet in O, the sides B_1C_1 and B_2C_2 must intersect in some point Z.

Now X, Y, and Z, lying respectively on the sides A_1B_1, A_1C_1, and B_1C_1, must lie in the plane of the triangle $A_1B_1C_1$, and for a similar reason they lie also in the plane of the tri-

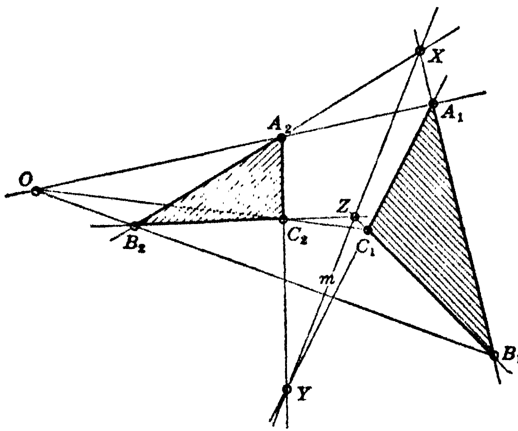


FIG. 10

angle $A_2B_2C_2$. They therefore lie on the common line of these two planes, which is a straight line.

Next, let us suppose that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ lie in the same plane and that the lines A_1A_2, B_1B_2, C_1C_2, joining pairs of corresponding vertices, intersect in one point O. To prove that in this case also the pairs of corresponding sides intersect in three points of a straight line, there are several methods which may be followed.

First, the theorem having been proved for the case in which the given triangles lie in different planes, the principle

of continuity [1] would suggest that, one of the planes being rotated about their common line and the theorem being true for every position of the rotating plane, it is true also for the position in which the planes coincide.

Or, second, the configuration for the theorem in which the given triangles lie in different planes, consists of ten lines and ten points, three points on each line and three lines through each point. If projected from a point not lying in any of the planes of the configuration, the projector will consist of ten rays and ten planes passing through the center of projection, three rays in each plane and three planes through each ray.

If the perspective triangles $A_1B_1C_1$ and $A_2B_2C_2$ lying in different planes (Fig. 10) are projected from a point P, they will yield three planes, PA_1A_2, PB_1B_2, PC_1C_2, intersecting in the ray PO, and three pairs of planes PA_1B_1 and PA_2B_2 PA_1C_1 and PA_2C_2, PB_1C_1 and PB_2C_2, intersecting in the rays PX, PY, PZ, respectively, which lie in a plane Pm determined by the point P and the line m on which X, Y, and Z lie. A section of this configuration by an arbitrary plane will consist of two triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ so situated that the lines joining pairs of corresponding vertices intersect in a point O' on the ray PO, while the pairs of corresponding sides intersect in points X', Y', Z', on the rays PX, PY, PZ, respectively. But these three rays lie in the plane Pm. Hence the points X', Y', Z', lie on one straight line, the intersection of the plane Pm and the arbitrary plane of section.

Or, third, a more direct method of proof is as follows.

Let it be given that the triangles $A_1B_1C_1$ and $A_2B_2C_2$,

[1] The principle of continuity (Poncelet, 1822) asserts that a property which is once demonstrated for a figure in one of its general forms and which remains true as the figure is changed continuously in accord with the conditions under which the property was first demonstrated, will be true also when the figure takes on a limiting form.

lying in one plane, are so situated that the lines A_1A_2, B_1B_2, and C_1C_2 meet in one point O. Through O take any straight line l not lying in the plane of the triangles (Fig. 11), and on this line choose any two points O_1 and O_2.

Since the line A_1A_2 passes through O and the line O_1O_2 also passes through O, the lines A_1O_1 and A_2O_2 lie in the same plane and consequently intersect in some point A'; similarly, B_1O_1 and B_2O_2 intersect in B', and C_1O_1 and C_2O_2 intersect in C'.

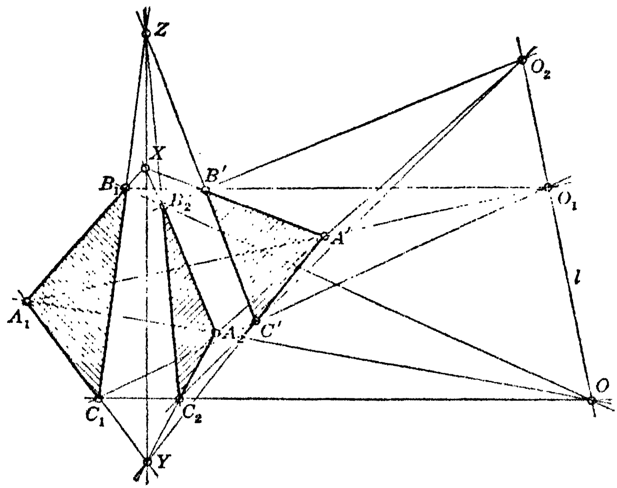


FIG. 11

The triangles $A_1B_1C_1$ and $A'B'C'$ lie in different planes and are so situated that the lines joining pairs of corresponding vertices A_1A', B_1B', C_1C', meet in the point O_1. Hence the pairs of corresponding sides A_1B_1 and $A'B'$, A_1C_1 and $A'C'$, B_1C_1 and $B'C'$, intersect in points of one straight line; namely, the line of intersection of the plane of the triangle $A'B'C'$ and the plane of the triangle $A_1B_1C_1$.

Similarly, the pairs of corresponding sides of the triangles $A_2B_2C_2$ and $A'B'C'$ intersect in points of this same line, since $A_1B_1C_1$ and $A_2B_2C_2$ lie in the same plane.

Consequently, A_1B_1 and A_2B_2, A_1C_1 and A_2C_2, B_1C_1 and B_2C_2, intersect in points of this line; namely, the points in which $A'B'$, $A'C'$, $B'C'$, meet this line.

25. Converse of Desargues' Theorem. The converse of Desargues' theorem as related to triangles in the same plane is the immediate reciprocal of the direct theorem and may be stated as follows.

THEOREM. *If two correlated triangles in the same plane are so situated that the pairs of corresponding sides intersect in points of a straight line, the lines joining the pairs of corresponding vertices will pass through one point.*

The proof of the converse theorem follows by reciprocation from the proof of the direct theorem, or the converse theorem may be accepted on the principle of duality.

Moreover, the proof of the direct theorem is seen to include the converse theorem if attention is given, for example, to the triangles A_1A_2Y and B_1B_2Z. In these triangles, lines joining pairs of homologous vertices intersect at the point X; in particular, the line YZ passes through X. The direct theorem states that in consequence of this relation, the intersections of pairs of homologous sides, namely, C_1, C_2, O, are on the same straight line. In other words, the points X, Y, Z being collinear, it follows that the line C_1C_2 passes through O, the intersection of A_1A_2 and B_1B_2.

Desargues' theorem and its converse are the fundamental theorems of projective pure geometry. From them as a starting point the whole subject will be developed.

26. Quadrangles and Quadrilaterals in Perspective.

THEOREM. *If two correlated complete quadrangles are so situ-*

THEOREM. *If two correlated complete quadrilaterals are so*

ated that five pairs of homologous sides intersect in points of one straight line, then the sixth pair of homologous sides will intersect in a point of the same line.

situated that five of the lines joining pairs of homologous vertices pass through one point, then the line joining the sixth pair of homologous vertices will pass through the same point.

To demonstrate the theorem on the left, let $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ (Fig. 12) be the two correlated quadrangles, either in the same plane or in different planes, and so situated that the pairs of homologous sides A_1B_1 and A_2B_2 intersect in P,

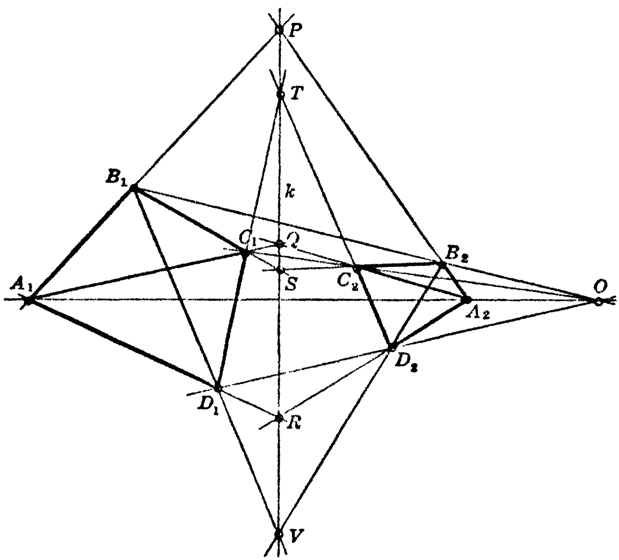


FIG. 12

A_1C_1 and A_2C_2 in Q, A_1D_1 and A_2D_2 in R, B_1C_1 and B_2C_2 in S, C_1D_1 and C_2D_2 in T, these five points, P, Q, R, S, T, lying on one line k. It is required to show that the sixth pair of homologous sides, B_1D_1 and B_2D_2 also intersect in a point of k.

Since in the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ the three pairs of homologous sides intersect in points of the line k, the lines A_1A_2, B_1B_2, C_1C_2, meet in one point O (§ 25). Similarly, in the triangles $A_1C_1D_1$ and $A_2C_2D_2$, the lines A_1A_2, C_1C_2, D_1D_2, meet in one point, and this point must coincide with O since in both cases it is the intersection of A_1A_2 and C_1C_2. Then, in the triangles $B_1C_1D_1$ and $B_2C_2D_2$, since the lines B_1B_2, C_1C_2, D_1D_2, meet in O, the pairs of homologous sides B_1C_1 and B_2C_2, B_1D_1 and B_2D_2, C_1D_1 and C_2D_2, must intersect in points of one straight line (§ 24). But this straight line is the line k, since by assumption two of these pairs of homologous sides intersect on k. That is to say, the sixth pair of homologous sides of the two quadrangles intersect at V on the same line as the first five.

The theorem on the right, which is the reciprocal of that on the left, may be proved analogously.

27. Harmonic Points on a Straight Line. In the preceding demonstration, if the points P and T of the line k should coincide at P (Fig. 13), and also the points S and R, at R, the argument and conclusion would be in no way affected. The homologous sides B_1D_1 and B_2D_2 would still intersect on the line k.

In this case, the points P and R of the line k are the intersections of pairs of opposite sides of the simple quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, while Q and V are points in each of which a pair of homologous diagonals of these simple quadrangles intersect.

Let us suppose that the simple quadrangle $A_1B_1C_1D_1$ has been drawn to fulfill the above conditions, that is, so that a pair of opposite sides intersect at P, the other pair at R, while one diagonal passes through Q, and the other diagonal through V. Then the second quadrangle, $A_2B_2C_2D_2$, may be constructed in any manner whatsoever, provided only that one

pair of opposite sides intersect at P, the other pair at R, and one diagonal passes through Q. The other diagonal will necessarily pass through V.

Thus, the point V on the line k has a fixed position relative to the three points P, Q, R, for it is the point in which the second diagonal of any simple quadrangle $A_1B_1C_1D_1$ intersects the line k, P and R being points of the line in which the pairs of opposite sides of the quadrangle intersect and Q the point through which the first diagonal passes.

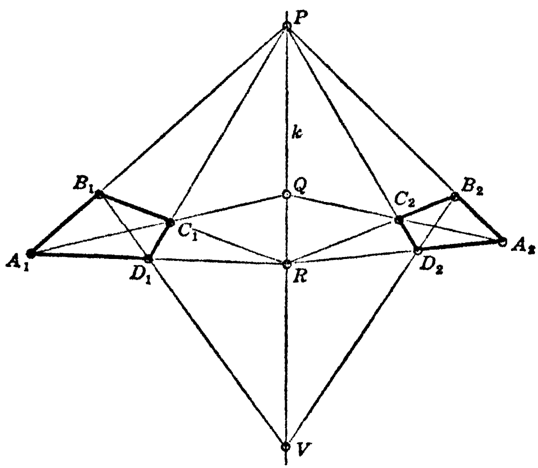


FIG. 13

Four points on a straight line related to each other as are these points P, Q, R, V are called ***harmonic points.***

DEFINITION. Four points on a straight line are said to be harmonic when they are so related that if two of them are the points of intersection of pairs of opposite sides of a simple quadrangle, the other two are the points of the line through which the diagonals of this quadrangle pass.[1]

[1] The harmonic relation of four points on a line was known to Apollonius (about 220 B.C.) and was studied by De la Hire (1640–1718) who gave the quadrilateral construction for the fourth harmonic point from three given points, as early as 1685.

It will be observed that in this definition the four points are grouped into two pairs, and when three of them are given, it being known which two of the three form a pair, the fourth point is determined.

For example, let A, B, C (Fig. 14) be three points of a harmonic set on a straight line, of which A and C are a pair.

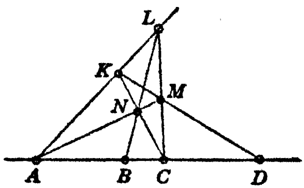


FIG. 14

In order to determine the fourth point D, which with B will form a pair in the harmonic set, we need only to construct a simple quadrangle $KLMN$ of which one pair of opposite sides intersect at A, the other pair at C, and one diagonal passes through B. The other diagonal will then intersect the line of A, B, C, at the required point D.

In this construction, three of the lines through the points, A, B, C, may be drawn at random and the figure is then fully determined.

28. The Pairs of Points in a Harmonic Set are Mutually Related. In the simple quadrangle $KLMN$ (Fig. 15) the points A and C on a given line are the intersections of pairs of opposite sides, and B and D are points on the diagonals. The four points, A, B, C, D, are therefore harmonic by definition.

Let KM and LN intersect at O. Draw the line AO intersecting the opposite sides, LM and KN, at Q and S, respectively, and CO intersecting the opposite sides, KL and MN, at P and R, respectively. Then $KPOS$ is a simple quadrangle of which one pair of opposite sides, KP and OS, intersect at A, the other pair, KS and PO, intersect at C, one diagonal, KO, passes through D, and consequently the other diagonal, PS, passes through B. Similarly, in the

simple quadrangles *OQMR*, *PLQO*, and *SORN*, one pair of opposite sides intersect at *A*, the other pair at *C*, one diagonal passes through *B*, and the other diagonal passes through *D*.

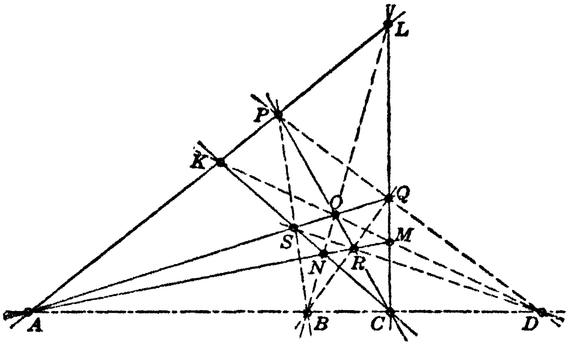


FIG. 15

The simple quadrangle *PQRS* is therefore such that one pair of opposite sides, *PS* and *QR*, intersect at *B*, the other pair, *PQ* and *RS*, intersect at *D*, one diagonal, *QS*, passes through *A*, and the other diagonal, *PR*, passes through *C*.

Hence, in the set of harmonic points *A*, *B*, *C*, *D*, the two pairs, *A*, *C*, and *B*, *D*, play exactly the same parts, for either pair may be the intersection points of opposite sides of a simple quadrangle while the other pair lie on the diagonals.

Of the four points, then, we may say that *A* and *C* are ***harmonic conjugates*** with respect to *B* and *D*; and likewise, *B* and *D* are harmonic conjugates with respect to *A* and *C*. Or we may say that *A* and *C* are harmonically separated by *B* and *D*; and, likewise, that *B* and *D* are harmonically separated by *A* and *C*.[1]

[1] Whenever mention is made of four points as harmonic, they will be so arranged that the first and third, second and fourth, are the pairs of conjugates.

29. Harmonic Conjugates Separate Each Other. That one pair of conjugates in a set of harmonic points on a straight line are actually separated by the other pair will appear from the following considerations.

Suppose, for the sake of argument, that in the harmonic set of points A, B, C, D (Fig. 16) the point A is not separated from its conjugate point C by the other two, but is separated from B by the other two. Then if the points A, B, C, D are projected from L on the line KM into the points K, O,

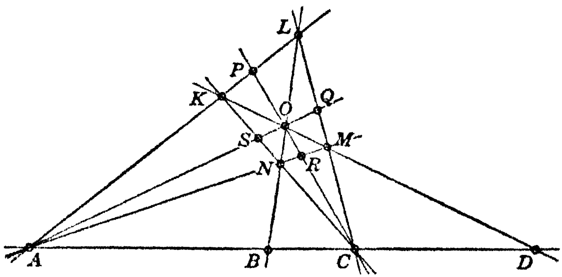


FIG. 16

M, D, respectively, the point K, which is the projection of A, would be separated from O by M and D. If the points are projected from N on the same line KM into the points M, O, K, D, respectively, the point M, which is the projection of A, would be separated from O by K and D. That is to say, O would be separated from K by M and D; and, at the same time, it would be separated from M by K and D, which is impossible (§ 9).

Hence, in a set of harmonic points on a straight line, two conjugate points are separated by the other two conjugate points.

30. If Two Points of a Harmonic Set Coincide, a Third Point Coincides with them. In Fig. 14, suppose the points A and C on the given line remain fixed, and also the points

K and L, so that the lines AK, CK, and CL remain fixed, while the point B is free to move along the line AC. If B moves to the right towards C, the line LB will rotate about L in the counter-clockwise sense; AM, which intersects LB on the fixed line CK, will rotate about A in the clockwise sense; and KM, which intersects AM on the fixed line CL, will rotate about K also in the clockwise sense, so that the point D will move towards C, along the line AC, to the left. If this motion is continued, the two points B and D will coincide with C at the same time. If B moves along AC in the opposite sense, D will also move in the opposite sense, and the two will coincide with A at the same time. If, then, the point B traverses the minor line-segment AC in either sense, its harmonic conjugate D relative to A and C will traverse the major line-segment AC in the opposite sense; and when two of the points coincide, a third will coincide with them no matter what position the fourth point may have on the line.

31. Pairs of Points Harmonically Separated by the same Third Pair. In the variable figure of § 30, if B_1, D_1 and B_2, D_2 are two pairs of corresponding positions of B and D, it is evident that these two pairs of points do not separate each other, and the following theorem may be stated.

THEOREM. *If two pairs of points on a straight line are both harmonically separated by the same third pair, they do not separate each other.*

It follows that two pairs of points on a straight line which separate each other cannot both be harmonically separated by the same third pair.

On the other hand, if two pairs of points on a straight line, P, Q, and R, S, do not separate each other (Fig. 17), there exists always a pair of points, M, N, which harmonically separate both these pairs.

For if a point M is chosen on the line-segment PQ which does not contain R and S, on the minor segment, say, its harmonic conjugate N_1 relative to P and Q lies on the major segment PQ; that is, on the segment which contains R and S. Also, the harmonic conjugate of M relative to R and S, N_2 say, lies on the line-segment RS which does not contain

FIG. 17

P and Q. If the point M traverses the minor segment PQ from P to Q, N_1 will traverse the major segment PQ in the opposite sense, while N_2 will traverse only a part of the minor segment RS; that is, only a part of the major segment PQ. At some point, therefore, in the segment RS which does not contain P and Q, the two points, N_1 and N_2 will coincide. Hence for some point M in the segment PQ which does not contain R and S, there is a corresponding point N in the segment RS which does not contain P and Q, such that M and N are harmonic conjugates relative to both P and Q, and R and S.

32. The Harmonic Relation is not Altered by Projection and Section. If the points and lines of Fig. 14 are projected from any point P outside the plane and a section is taken of the projecting figure, a new quadrangle $K'L'M'N'$ is obtained in the plane of section, and a new set of points $A'B'C'D'$ is obtained on the rays PA, PB, PC, PD, so related that pairs of opposite sides of the quadrangle $K'L'M'N'$ intersect in A' and C', while the diagonals pass through B' and D', respectively. The points A', B', C', D' therefore are harmonic and we have the following theorem.

THEOREM. *If four harmonic points on a straight line are projected from any point not on the line, a section of the projecting rays is again a set of harmonic points.*

In other words, the harmonic relation of four points on a straight line is not altered by projection.

DEFINITION. If a set of harmonic points on a straight line is projected from any point not on the line, the rays so drawn are called *harmonic rays*.

Hence any section of a set of harmonic rays is a set of harmonic points.

Since to any point on a straight line there is only one harmonic conjugate relative to two fixed points on the line (§ 27), it follows that if there are given two sets of harmonic points on a line in which a pair of conjugate points in one set coincide with a pair of conjugates in the other, and a third point in the one coincides with a third point in the other, then the fourth point in the one set coincides with the fourth point in the other. Also, if two sets of harmonic rays have the same center and three rays of one set coincide with three rays of the other, conjugates with conjugates, the fourth rays of the two sets must coincide.

33. Harmonic Properties of the Complete Quadrangle and Complete Quadrilateral. Thus far it is the simple quadrangle with vertices K, L, M, N, taken in that order only, which has been considered. These vertices may, however, be taken in the order K, L, N, M, or in the order K, N, L, M, and in either case a different simple quadrangle is obtained (§ 20).

If the vertices are taken in the order K, L, N, M, the lines KL and NM, intersecting at A, are opposite sides (Fig. 16), as are also KM and LN, intersecting at O; KN and LM are diagonals intersecting the line AO at S and Q, respectively. Therefore the points A, S, O, Q are harmonic by definition (§ 27).

Similarly, by taking the vertices in the order K, N, L, M, it follows that the points C, P, O, R are harmonic by definition.

Also, since the four rays through O project the harmonic points A, B, C, D, they are harmonic rays and the points A, N, R, M, as well as the points A, L, P, K, being sections of these four rays in order, are likewise harmonic. Similarly, Q, L, C, M and S, N, C, K are harmonic, and by projecting these points from A, we see that O, L, B, N and O, K, D, M are likewise harmonic. In fact, in Fig. 16 wherever four points occur on a line, or four lines pass through a point, they are harmonic.

Many of these results are contained in the following reciprocal theorems.

THEOREM. *In a complete quadrangle the sides intersecting in one diagonal point are harmonically separated by the lines through that point projecting the other diagonal points.*

THEOREM. *In a complete quadrilateral the vertices lying on one diagonal are harmonically separated by the points of intersection of this diagonal with the other two diagonals.*[1]

34. The Harmonic Relation is not Altered by Permutation. From the definition of harmonic points on a straight line (§ 27) it should be noted that not only are the points A, B, C, D harmonic (Fig. 16), but so also are C, B, A, D, since in the first and third of these points, pairs of opposite sides of the quadrangle intersect, while through the second and fourth the diagonals pass. Moreover, as has already been noted, the points B and D may be interchanged with A and C since the two pairs play the same parts in the harmonic set. Hence we may state the following theorem.

THEOREM. *If A, B, C, D are a set of harmonic points on a straight line, any arrangement of these points in which the pairs are not broken will likewise be harmonic.*

That is to say, not only are the points A, B, C, D, taken

[1] Carnot, *Geométrie de position*, 1803.

in that order, harmonic, but any permutation of the four points is a harmonic set if only the cycle A, B, C, D is not broken.[1]

EXERCISES

1. Given three rays of a pencil, to find the fourth ray which is the harmonic conjugate of a particular one of the three, relative to the other two.

2. Through a given point in a plane draw a line which will pass through the inaccessible point of intersection of two given lines in the same plane.

3. Defining harmonic planes of a pencil as four planes of which a section is four harmonic rays, find the fourth harmonic plane to three given planes of a pencil, the required plane to be conjugate to a particular one of the three.

4. State the space reciprocal of Desargues' theorem as related to triangles not in the same plane.

5. A line is drawn through a fixed point A to intersect two given fixed planes β and γ in points B and C, respectively. What is the locus of D, the harmonic conjugate of A relative to B and C?

6. Prove that in a complete quadrangle the sides of the diagonal triangle intersect the sides of the quadrangle in six points other than the vertices of the triangle, which lie by threes on four straight lines (see Exercise 8, p. 23).

7. If each of two points is harmonically separated from a third given point by a pair of opposite edges of a tetrahedron, the two points are harmonically separated from each other by the third pair of opposite edges.

SUGGESTION. The plane of the three given points intersects the tetrahedron in a complete quadrilateral whose diagonals intersect in the three points.

8. A straight line intersects the sides of a triangle ABC in the points A_1, B_1, C_1, and the harmonic conjugates, A_2, B_2, C_2, of these points are determined relative to the two vertices on the same side, so that A_1, B, A_2, C, for example, are harmonic. Show that A_1, B_2, C_2; B_1, C_2, A_2; C_1, A_2, B_2, are collinear, and that AA_2, BB_2, CC_2 are concurrent, as are also AA_2, BB_1, CC_1; AA_1, BB_2, CC_1; and AA_1, BB_1, CC_2.

[1] Reye, *Geometrie der Lage*, 1866.

CHAPTER IV

METRIC PROPERTIES

35. Introduction of Metric Relations. By the methods of elementary geometry it is readily shown that the diagonals of a parallelogram bisect each other; in other words, that the point of intersection of the two diagonals is the mid-point of each. In Fig. 16, if the line AC moves out so as to become the infinitely distant line of the plane, the quadrangle $KLMN$ becomes a parallelogram (Fig. 18), the points B

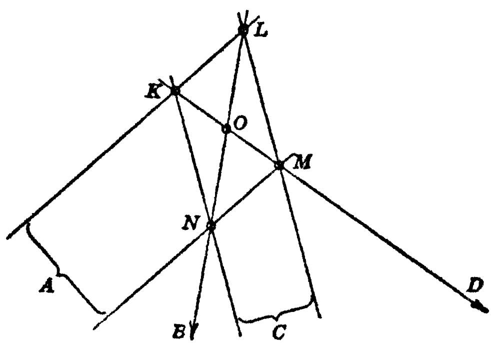


FIG. 18

and D are infinitely distant, and the point O is the mid-point of both the line-segments KM and LN. But the points K, O, M, D are harmonic, as are also L, O, N, B (§ 33); that is, B is the harmonic conjugate of O relative to L and N, and D is the harmonic conjugate of O relative to K and M. Consequently, we may state the following theorem.

THEOREM. *The harmonic conjugate of the mid-point of any line-segment with respect to the end points of that segment is*

the point at infinity on the line; and conversely, of four harmonic points on a line, if one is infinitely distant, its conjugate is the mid-point of the line-segment determined by the other two.

If, then, we are given a line-segment and its mid-point, we are able to draw a line through any given point parallel to the segment; and, conversely, if we are given a line-segment and a line parallel to it, we can find the mid-point of the segment.

First, let AC be a given line-segment, B its mid-point, and P a given point through which a parallel is to be drawn (Fig. 19). Projecting the points A, B, C, from P and taking any section A', B', C', of the rays so drawn, the harmonic conjugate, D', of the point B' relative to A' and C', which may be found by the quadrangle construction, determines a line through P parallel to AC. For, since the rays $P(A', B', C', D')$ are harmonic, the section of them by the line AC is harmonic; and, since B is the mid-point of the segment AC, its harmonic conjugate is infinitely distant. Hence the line PD' intersects AC in its infinitely distant point, or is parallel to AC.

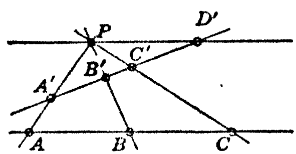


FIG. 19

Conversely, given a line-segment AC and a line parallel to it, by projecting the end points of the line-segment from any point of the parallel line, a similar construction will determine the mid-point of the segment.

Since the center of a circle is the mid-point of any diameter, if we are granted the use of circles as in elementary geometry, it is always possible to draw a line-segment with a given mid-point. Consequently, granted the use of circles, we can draw through a given point a line parallel to any given line. For, by the use of a circle, a line-segment and

its mid-point may be determined on the given line, and then a parallel can be drawn through the given point.

36. Ratios of the Line-Segments Determined by Harmonic Points. Suppose A, B, C, D are four harmonic points on a straight line (Fig. 20), which are projected from any point P by four harmonic rays. If through B a line is drawn parallel to the ray PD, cutting PA at A' and PC at C', the point B will be the mid-point of the segment $A'C'$, since its harmonic conjugate, relative to A' and C', is infinitely distant.

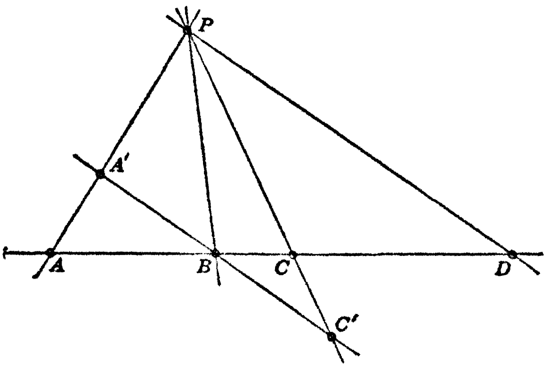


Fig. 20

The triangle $AA'B$ is similar to the triangle APD, and therefore

$$\frac{AB}{BA'} = \frac{AD}{DP}.$$

Also, the triangle $BC'C$ is similar to the triangle DPC, and therefore

$$\frac{BC}{BC'} = \frac{DC}{DP}.$$

Combining these results and remembering that BA' equals BC' we have the relation

$$\frac{AB}{BC} = \frac{AD}{DC},$$

or, the segment AC is divided internally and externally in the same ratio at the points B and D.

This equality of ratios may be written in the order

$$\frac{BA}{AD} = \frac{BC}{CD}.$$

From this it is seen that the segment BD is likewise divided internally and externally in the same ratio at the points A and C. In other words, the points A, C, and the points B, D, play the same parts in the harmonic set, as was pointed out in § 28.

37. Metric Formulas. (1) If we have regard to the sense in which a line-segment is measured, so that, for example, the segment AB is the negative of the segment BA, the above equality of ratios takes the form

(A) $$\frac{AB}{BC} = -\frac{AD}{DC},$$

since the segment DC (Fig. 20) is measured in the sense opposite to that of the other three. It is in this form that the equality of ratios will ordinarily be stated.

(2) The fractions in formula (A) being inverted, we have the relation,

$$\frac{BC}{AB} = -\frac{DC}{AD}; \qquad \text{or} \qquad \frac{BA + AC}{AB} = -\frac{DA + AC}{AD}$$

from which we find that

$$\frac{AC}{AB} - 1 = 1 - \frac{AC}{AD}.$$

Transposing terms and dividing by AC, we have the formula

(B) $$\frac{1}{AB} + \frac{1}{AD} = \frac{2}{AC}.$$

(3) Again, if M is the mid-point of the segment BD, the formula

$$\frac{AB}{BC} = -\frac{AD}{DC}$$

may be written in the form

$$\frac{AM + MB}{BM + MC} = -\frac{AM + MD}{DM + MC}.$$

Simplifying and remembering that $BM = MD$, we have the formula

(C) $$AM \cdot CM = BM^2.$$

The formulas (A), (B), and (C) are of frequent use in the metric theory of harmonics.

Formula (B) may be written in the form

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{AC} - \frac{1}{AD},$$

from which it is seen at once that the line segments AB, AC, and AD are in harmonic progression since their reciprocals are in arithmetic progression. This identifies the harmonic relation in geometry with that of algebra.

38. Conjugate Rays at Right Angles. In Fig. 20, if B were the mid-point of AC, the point D would be at infinity, and the ray PD would be parallel to AC. If, in addition, P were so chosen that APC is isosceles, PB and PD would be at right angles and would bisect the angles formed by the rays PA and PC.

Hence we have the following theorem.

THEOREM. *If four rays through a point are harmonic and one pair of conjugate rays are at right angles, they bisect the angles formed by the other pair; and conversely, if two rays at right angles make equal angles with two others through their point of intersection, the four rays are harmonic.*

39. Orthogonal Circles determine Harmonic Points on a Diameter. If A, B, C, D are harmonic points on a line and M is the mid-point of the segment BD, it is clear that A and C are on the same side of M since $AM \cdot CM = BM^2$ (§ 37), and that consequently M lies outside of any circle through A and C.

The tangent from M to any circle k through A and C (Fig. 21) is equal in length to the segment MB or MD (§ 37), and if the circle l on BD as diameter is drawn, these two circles k and l will intersect orthogonally, the tangent to one at a point of intersection being a radius of the other.

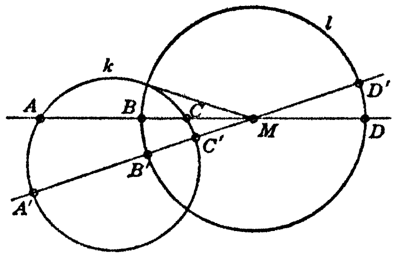


Fig. 21

Moreover, any diameter of the circle l intersecting that circle at B' and D' and the circle k at A' and C' is cut harmonically. For, $MB' = MB$ and $MA' \cdot MC' = MA \cdot MC$. Therefore $MA' \cdot MC' = MB'^2$.

Hence the following theorem may be stated.

Theorem. *If four points A, B, C, D on a straight line are harmonic, the circle having as diameter the segment determined by one pair of conjugate points will cut orthogonally any circle through the other pair of conjugate points; and, conversely, if two circles intersect orthogonally they cut a diameter of either of them in four harmonic points.*[1]

40. Construction for Points Harmonically Separating Two Given Pairs. Formula C, § 37, lends itself readily to a solution of the problem considered in § 31.

[1] Poncelet, *Propriétés projectives des figures*, 1822.

PROBLEM. *Given two pairs of points, A_1, B_1 and A_2, B_2, on a straight line, to find a pair of points on the line which will harmonically separate both given pairs.*

If any point P is taken, not lying on the given line (Fig. 22), and circles PA_1B_1, PA_2B_2, are drawn they will intersect a second time at some point Q, and if the line PQ intersects the given line at M, we have the relation,

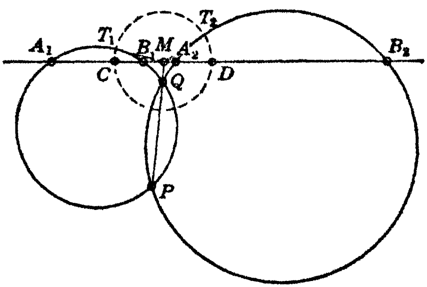


FIG. 22

$$MA_1 \cdot MB_1 = MP \cdot MQ$$
$$= MA_2 \cdot MB_2.$$

If tangents MT_1 and MT_2 are drawn from M to the two circles, they are equal since $MT_1^2 = MP \cdot MQ = MT_2^2$, and a circle with center M and radius MT_1 or MT_2 will cut the given line at points C and D such that

$$MC^2 = MA_1 \cdot MB_1 = MA_2 \cdot MB_2 = MD^2.$$

Hence C and D harmonically separate both A_1, B_1 and A_2, B_2.

If the pairs of points A_1, B_1 and A_2, B_2 separate each other on the given line, the two circles through P will intersect a second time on the side of the given line opposite to P and the point M will lie inside both circles; hence no tangents from M can be drawn, and consequently no points C and D can be found, as was pointed out in § 31.

41. Anharmonic Ratios or Cross-Ratios. If A, B, C, D are four points on a straight line, the line-segment determined by any two of them, A and D, say, is divided into other segments by the remaining two points, B and C. If, now, we take the ratios of these segments, AB/BD and

AC/CD, and form the ratio of these ratios; that is, form the ratio,

$$\frac{AB}{BD}\Big/\frac{AC}{CD}, \qquad \text{or} \qquad \frac{AB\cdot CD}{AC\cdot BD},$$

this double ratio is called an ***anharmonic ratio*** or a ***cross-ratio*** of the four points.[1]

If the cross-ratio

$$\frac{AB\cdot CD}{AC\cdot BD}$$

is written in the equivalent form

$$\frac{BA\cdot DC}{BD\cdot AC},$$

it will be seen that the two pairs of points A, D, and B, C, play exactly the same parts in forming the cross-ratio; so that instead of taking the line segment AD, divided at B and C, we may choose, as well, the line-segment BC, divided at A and D, and proceeding in the way described, we should obtain the same result as before.

42. Six Different Cross-Ratios for the same Four Points. For the same four points A, B, C, D, on a line, there are six different expressions which satisfy the definition of a cross-ratio of the points. For the points may be paired in three different ways to determine the line-segment; namely, A with B, A with C, and A with D, the other two points in each case being the points of division, and the ratio of the two ratios, as AC/CB and AD/DB, may be taken in either order.

[1] The term "anharmonic ratio" was used by Chasles (1793–1880) who developed the relation denoted by it as an instrument of great value in modern pure geometry. The term "cross-ratio" is due to Professor W. K. Clifford of London (1845–1879). Cross-ratio is more suggestive of the manner in which the double ratio is formed and is in common use as the equivalent of the older term.

The six cross-ratios of the four points, therefore, are the fractions

$$\frac{AC\cdot DB}{AD\cdot CB}, \quad \frac{AD\cdot BC}{AB\cdot DC}, \quad \frac{AB\cdot CD}{AC\cdot BD},$$

and their reciprocals. In the first of these ratios, A and B determine the line-segment of which C and D are points of division, while in the second, AC is the segment, and in the third, AD is the segment, the other two points in each case being the points of division.

By making use of a well-known relation among four points, A, B, C, D, on a line, namely,

$$AB\cdot CD + AC\cdot DB + AD\cdot BC = 0,\text{[1]}$$

and assuming

$$\frac{AB\cdot CD}{AC\cdot BD} = \lambda$$

we can readily show that the six cross-ratios of the same four points have the values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda-1}{\lambda}, \quad \frac{\lambda}{\lambda-1}.$$

43. Cross-Ratios are Unaltered by Projection. If the points A, B, C, D, on a fixed line are projected from any point P by the rays a, b, c, d (Fig. 23) and if h is the distance of P from the fixed line, we have the following relation.

Twice the area of the triangle APB equals

$$h\cdot AB = PA\cdot PB\cdot \sin (ab),$$

where sin (ab) signifies the sine of the angle made by the rays a and b. Similarly,

$$h\cdot CD = PC\cdot PD\cdot \sin (cd),$$
$$h\cdot AC = PA\cdot PC\cdot \sin (ac),$$
$$h\cdot BD = PB\cdot PD\cdot \sin (bd).$$

[1] This relation may be proved readily by writing $AB = AO + OB$; $CD = CO + OD$; etc., where O is any fifth point on the line.

Therefore the cross-ratio

$$\frac{AB \cdot CD}{AC \cdot BD} = \frac{\sin(ab) \cdot \sin(cd)}{\sin(ac) \cdot \sin(bd)},$$

a quantity which depends only on the angles formed by the rays a, b, c, d. An arbitrary section, A', B', C', D', of the

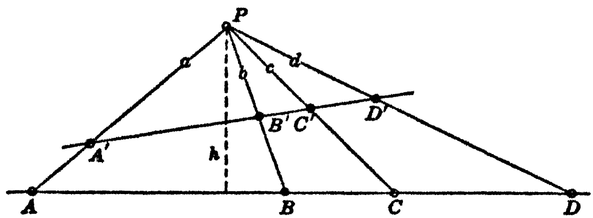


FIG. 23

rays a, b, c, d, will therefore have the same cross-ratios as the points A, B, C, D. The fraction

$$\frac{\sin(ab) \cdot \sin(cd)}{\sin(ac) \cdot \sin(bd)},$$

is defined as a cross-ratio of the four rays through P, and other cross-ratios of the four rays may be formed similarly. Moreover, a different point P_1 projecting A, B, C, D, by the rays a_1, b_1, c_1, d_1, of which A_1, B_1, C_1, D_1, is an arbitrary section, will yield the following relations.

$$\frac{AB \cdot CD}{AC \cdot BD} = \frac{\sin(a_1b_1) \cdot \sin(c_1d_1)}{\sin(a_1c_1) \cdot \sin(b_1d_1)} = \frac{A_1B_1 \cdot C_1D_1}{A_1C_1 \cdot B_1D_1}.$$

Hence we have the following theorem.

THEOREM. *The cross-ratio of four points on a line, or of four rays through a point, is unaltered by projection.*

The cross-ratio of four planes of an axial pencil is defined as the cross-ratio of the four rays in which the pencil is cut by an arbitrary plane. This cross-ratio is readily seen to be independent of the position of the plane and is likewise unaltered by projection.

44. Cross-Ratios are Unaltered by Permutation. If A, B, C, D are four points on a line, their cross-ratio formed in any one of the six ways is unaltered no matter in what order the points are taken, provided that when any two of them are interchanged the other two are also interchanged.

For example, if the cross-ratio of the given points taken in the order A, B, C, D, is written in the form

$$\frac{AB\cdot CD}{AC\cdot BD},$$

the similar cross-ratio of the points in the order B, A, D, C, in which A and B are interchanged and also C and D, is

$$\frac{BA\cdot DC}{BD\cdot AC};$$

but this fraction equals

$$\frac{AB\cdot CD}{AC\cdot BD}.$$

45. The Cross-Ratio of Four Harmonic Points. If it should happen that the cross-ratio

$$\frac{AB\cdot CD}{AC\cdot BD}$$

of four points A, B, C, D is equal to -1, it will be readily seen that the points are a harmonic set in which A and D, B and C, are conjugate points. For, if the cross-ratio is equal to -1, we shall have

$$\frac{AB\cdot CD}{AC\cdot BD} = -1,$$

from which it follows that

$$AB/BD = -AC/CD,$$

and the line-segment AD is divided internally and externally at B and C in the same ratio. In this case, the pairs of points A, D and B, C, must separate each other on the line.

EXERCISES

1. In a plane there are given a line-segment AC and a parallelogram. Without the use of circles, find the mid-point of AC and through a given point draw a line parallel to AC.

2. If A, B, C, D are four harmonic points on a straight line and a circle of which S is any point is described on AC as diameter, prove that the arc intercepted between the rays BS and DS, or these rays produced, is bisected at A or at C.

3. If A, B, C, D, any four points on a circle, are projected from the points P and Q, also on the circle, show that the cross-ratio of the rays $P\ (A, B, C, D)$ is equal to the like cross-ratio of the rays $Q\ (A, B, C, D)$.

4. If the bisector of the angle A of a triangle ABC meets the opposite side at A_1 and perpendiculars from B and C on AA_1 meet that line at B_1 and C_1, respectively, show that the points $AB_1A_1C_1$ are harmonic.

5. A circle is circumscribed about a square and the vertices of the square are projected from any point of the circle. Prove that the four projecting rays are harmonic.

6. A chord AC of a circle is perpendicular to a diameter BD. Show that the rays projecting the points A, B, C, D from any point P of the circle are harmonic.

7. Through a given point P in the plane of two given intersecting lines a and b draw a line intersecting the given lines at A and B, respectively, so that (1) $PA = PB$; and (2) $PA = AB$.

8. If A, B, C, D are harmonic points on a straight line, prove that

$$\frac{1}{BA} + \frac{1}{BC} = \frac{2}{BD}.$$

9. If four points on a straight line are harmonic, show that their cross-ratio equals -1, $\frac{1}{2}$, or 2, according as the cross-ratio is formed in one order or in another.

10. Two tangents TP and TQ are drawn to a circle of which PR is a diameter. If QN is drawn perpendicular to PR, prove that the rays $Q(T, P, N, R)$ are harmonic.

11. Through a fixed point O two straight lines are drawn meeting two fixed lines in the plane at A, B, and C, D, respectively. Show that the locus of the intersection of AD and BC is a straight line.

12. Four points, A, B, C, D, are given in order on a straight line and the points M and N harmonically separate both A, D, and B, C. If O is the mid-point of the line-segment MN and P is any point of the circle on MN as diameter, show that OP is tangent to both the circles APD and BPC, and that the angles APB and CPD are equal. In other words, show that the circle on MN as diameter is the locus of points at which the segments AB and CD subtend equal angles.

How may the points be arranged in order on the line so as to make the angles supplementary instead of equal?

13. If a, b, c, d are four rays through a point intersected by a straight line in the points A, B, C, D, and a line is drawn through B parallel to the ray c intersecting a and d in the points A' and D', respectively, prove that the cross-ratio $AB \cdot CD \,/\, AC \cdot BD$ is equal to the ratio $A'B \,/\, D'B$.

14. Making use of the result in Exercise 13, show how to find, by a geometric construction, the fourth point D of a straight line on which three points, A, B, C, are given, such that the cross-ratio $AB \cdot CD \,/\, AC \cdot BD$ may have a given value.

15. If A, B, C, D are any four points in order on a straight line and circles are described on the segments AC and BD as diameters intersecting at P, show that the cross-ratio $AB \cdot CD \,/\, AC \cdot BD$ equals $\sin^2 \theta$, where θ is one-half the angle between radii of the circles drawn from P.

Show also that the other five cross-ratios of the four points may be expressed as functions of the angle θ.

CHAPTER V

PROJECTIVELY RELATED PRIMITIVE FORMS

46. Correlation of Geometric Primitive Forms. At the beginning of Chapter III, rectilinear figures were correlated to each other by relating each vertex of the one to a particular vertex of the other and, as a consequence, each side of the one to a particular side of the other. Vertices and sides so related were said to be homologous.

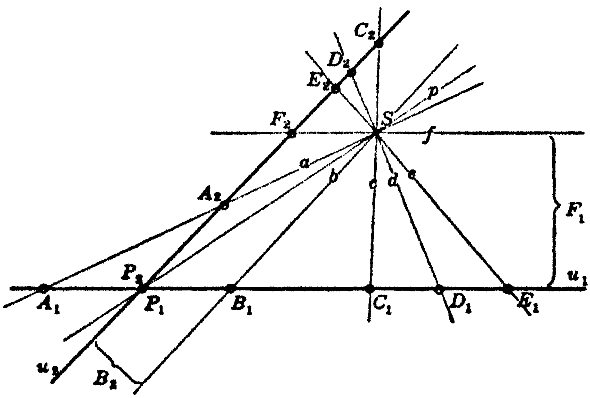


FIG. 24

Likewise, two primitive forms of the same order may be correlated by associating each element of the one with a definite element of the other. Such a correlation may be established in many ways, but most simply perhaps by the method of projection and section. As in the case of rectilinear figures two associated elements in correlated forms are said to correspond or to be homologous.

If two ranges of points, for example, are sections of the same pencil of rays (Fig. 24), each element of one range may be correlated to that element of the other which lies on the same ray of the pencil. By this means there is established a point-to-point relation between the two ranges; to every point, A_1, B_1, C_1, $\cdots$ of the one range there is a definite corresponding point, A_2, B_2, C_2, $\cdots$ of the other. In particular, the ideal point of the one range corresponds to an actual point of the other, unless by chance the two ranges are on parallel lines, in which case the ideal point of the one range corresponds to the ideal point of the other.

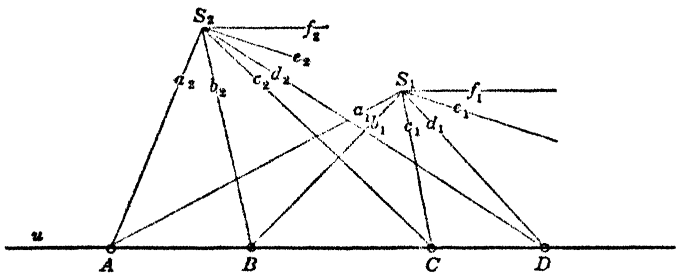


FIG. 25

On the other hand, if a range of points is projected from two centers by two pencils of rays (Fig. 25), these pencils may be correlated by considering those rays homologous which project the same point of the range.

47. Primitive Forms of the Same Kind in Perspective. DEFINITION. If two primitive forms of the same kind are projections or sections of the same third form of a different kind and those elements of the two forms are correlated which project or which lie on the same element of the third form, the two forms are said to be ***perspective*** to each other, or to be ***in perspective relation.***

In Fig. 24, the two ranges of points u_1 and u_2 are perspective to each other if the points A_1, B_1, C_1, $\cdots$ of u_1 are correlated to the points A_2, B_2, C_2, $\cdots$ of u_2, respectively, since when so correlated pairs of corresponding points lie on the same ray of the pencil S.

In Fig. 25, the two pencils of rays S_1 and S_2 are perspective to each other if the rays a_1, b_1, c_1, $\cdots$ of S_1 are correlated to the rays a_2, b_2, c_2, $\cdots$ of S_2, since when so correlated pairs of corresponding rays project the same point of the range u.

Similarly, two pencils of rays are perspective when they are sections of the same pencil of planes and those rays are correlated which lie in the same plane of the pencil, and two pencils of planes are perspective when they project the same pencil of rays or the same range of points and those planes are correlated which project the same ray of the pencil or the same point of the range.

48. The Common Element of Perspective Forms is Self-Corresponding.

THEOREM. *If two primitive forms of the same kind are perspective, their common element, if any, is self-corresponding.*

By ***self-corresponding*** we mean that the element considered as belonging to one of the two forms is homologous to itself when considered as belonging to the other form. Thus, in Fig. 24, the point P_1 of u_1 is the same as P_2 of u_2, and since P_1 and P_2 lie on the same ray of the pencil S they are homologous elements. That is, the common point of u_1 and u_2 is self-corresponding.

Similarly, the common ray of the pencils S_1 and S_2 (Fig. 25) is self-corresponding since, whether considered as a ray of S_1 or of S_2, it projects the same point of u. Also, in two perspective pencils of planes projecting the same pencil of rays the common plane is self-corresponding.

Two perspective pencils of rays may have no common ray as, for example, when they are sections of the same pencil of planes and do not lie in the same plane. So also two perspective ranges of points which lie in the same pencil of planes may have no common point.

49. Primitive Forms of Different Kinds in Perspective. DEFINITION. Two primitive forms of different kinds are said to be perspective when one of them is a projection or a section of the other and the elements of the one form are correlated to those elements of the other on which they lie or which they project.

In Fig. 24, the range of points u_1 is perspective to the pencil of rays S if the points A_1, B_1, C_1, $\cdots$ of u_1 are correlated to the rays a, b, c, $\cdots$ of S, and similarly the range of points u_2 is perspective to the pencil of rays S if the points of u_2 are correlated to the rays of S on which they lie. In Fig. 25, both pencils of rays S_1 and S_2 are perspective to the range of points u if the rays of the pencils are correlated to the points of the range through which they pass.

50. Projectivity in Primitive Forms. DEFINITION. If two primitive forms of the first order are so correlated that to every set of harmonic elements in the one form there corresponds a set of harmonic elements in the other, the two forms are said to be correlated ***projectively,*** or to be ***projective.***[1]

It will be recalled that by a primitive form of the first order is meant a range of points, a pencil of rays, or a pencil of planes. If two such forms are related perspectively (§§ 47, 49), they are evidently correlated projectively, since in perspective forms any set of harmonic elements in the one corresponds to a set of harmonic elements in the other.

[1] This definition of the projective relation between two forms is due to Von Staudt (*Geometrie der Lage*, Nürnberg, 1847).

It follows also that if two forms are each projective to the same third form they are projective to each other.

51. Sequence of Homologous Elements in Projective Forms. From the definition of projectively related forms we may deduce the following theorem which is essential to much that follows.

THEOREM. *If two primitive forms of the first order are projective, a sequence of elements in the one form corresponds to a similar sequence of elements in the other.*

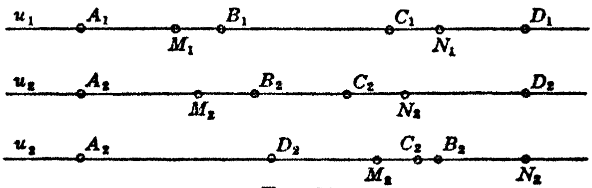


FIG. 26

To establish this theorem we need only to consider two projective ranges of points u_1 and u_2 (Fig. 26), since all other cases may be reduced to this one. In these two forms let the points A_1, B_1, C_1, D_1 of u_1 correspond to the points A_2, B_2, C_2, D_2 of u_2. We shall show, first, that if A_1 and B_1 in the first range are not separated by C_1 and D_1, then in the second range A_2 and B_2 cannot be separated by C_2 and D_2.

If A_1 and B_1 are not separated by C_1 and D_1, there is a pair of points M_1, N_1, which harmonically separate both of these pairs (§ 31), so that A_1, M_1, B_1, N_1 are a harmonic set, as are also M_1, C_1, N_1, D_1.

Since the two forms u_1 and u_2 are projective, and by definition a set of harmonic elements in one range corresponds to a set of harmonic elements in the other, the points M_2 and N_2 in u_2, corresponding to M_1 and N_1 in u_1, are such that A_2, M_2, B_2, N_2 are a harmonic set, as are also M_2, C_2, N_2, D_2. In other words, M_2, N_2 harmonically separate both A_2, B_2

and C_2, D_2. Therefore A_2 and B_2 are not separated by C_2 and D_2 (§ 31).

If, then, P_1, Q_1, R_1, S_1, T_1, V_1, $\cdots$ is any sequence of points in u_1 so related that no two neighboring points are separated by the point named just before and the point named just after them, the points P_2, Q_2, R_2, S_2, T_2, V_2, $\cdots$ corresponding to them, respectively, in u_2, must be such that no two neighboring points are separated by the point named just before and the point named just after them.

Suppose, now, P_1, Q_1, R_1, S_1, $\cdots$ are consecutive points[1] of the range u_1, the corresponding points P_2, Q_2, R_2, S_2, $\cdots$ must also be consecutive points of the range u_2. For, if Q_2 and R_2, for example, are not consecutive, there is at least one point K_2 which with some other point L_2 will separate them. If Q_2 is separated from R_2 by K_2 and L_2, it is not separated from either K_2 or L_2 by the remaining two points (§ 9). Consequently, of the points Q_1, K_1, R_1, L_1, of the range u_1, Q_1 is not separated from either K_1 or L_1 by the remaining two; therefore it is separated from R_1 by the remaining two, in which case Q_1 and R_1 cannot be consecutive points as was assumed. Hence a series of consecutive points in the one form corresponds to a series of consecutive points in the other.

Similar considerations applied to other sequences will yield similar results.

52. Superposed Projective Forms. If two pencils of rays, S_1 and S_2, having different centers, lie in the same plane, any straight line u of the plane passing through neither of the centers will cut each pencil in a range of points. There will thus lie on the line u two ranges of points, u_1 and u_2, and if the two pencils of rays are correlated in any manner, the ranges of points will be similarly correlated.

[1] Two points on a line are defined to be consecutive when they are so situated that there is no point which with another will separate them.

If the two pencils are related projectively as in Fig. 27, both pencils being perspective to the same range of points A, B, C, D, $\cdots$ (§ 50), and the rays a_1, b_1, c_1, d_1, $\cdots$ of S_1 being homologous, respectively, to the rays a_2, b_2, c_2, d_2, $\cdots$ of S_2, then the ranges of points u_1 and u_2, which are sections of these pencils and lie on the line u, are correlated projectively, and the points, A_1, B_1, C_1, D_1, $\cdots$ of u_1 cor-

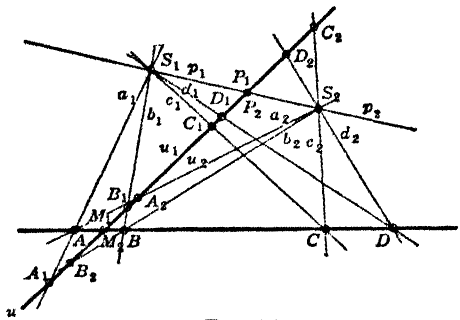


FIG. 27

respond, respectively, to the points A_2, B_2, C_2, D_2, $\cdots$ of u_2. Every point of the line u is an element of the range u_1 and also an element of the range u_2. On the line u, then, there are two ***superposed*** ranges of points, projectively related, and each point of the line must be thought of as belonging both to the one range and to the other.

Similarly, if two projectively related ranges of points u_1 and u_2 lie in the same plane and are both projected from a point S of the plane (Fig. 28), there will be constructed with the same center S, two projectively related pencils of rays in which every ray through S must be thought of as belonging both to the one pencil and to the other. In other words, with the common center S, there are two superposed pencils of rays lying in the same plane which are projectively related.

53. Self-Corresponding Elements in Superposed Projective Forms. In two superposed projective pencils of

rays (Fig. 28) it may happen that the same ray through S projects both the point M_1 of the range u_1 and its corresponding point M_2 of the range u_2. This ray counted as belonging to the one pencil would then correspond to itself in the other pencil, and it is therefore a self-corresponding ray in the two projectively related pencils.

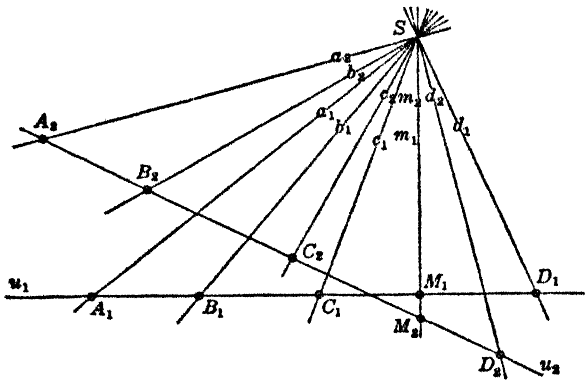


FIG. 28

In the same way, it may happen that the line u (Fig. 27) cuts a pair of corresponding rays m_1 and m_2 of the projectively related pencils, S_1 and S_2, at the same point M, in which case M is a self-corresponding point of the superposed projective ranges.

The center S of the two superposed pencils of rays projecting the ranges u_1 and u_2 (Fig. 28) may be the point of intersection of rays joining two pairs of homologous points, A_1 and A_2, B_1 and B_2, of the projective ranges u_1 and u_2 (Fig. 29), in which case the superposed projective pencils of rays will have two self-corresponding rays while other rays are not self-corresponding. So also, the line u of two superposed ranges of points may pass through the points of intersection of two pairs of homologous rays a_1 and a_2, b_1 and

b_2, of the projective pencils S_1 and S_2 (Fig. 30), in which case the superposed projective ranges of points will have two self-corresponding points while other points are not self-corresponding.

Since any two one-dimensional primitive forms are related projectively to each other if either is projective to a section or a projection of the other (§ 50), it follows that if a pencil of rays is projective to a range of points, two rays of the pencil may pass through the points of the range homologous to them while others do not; and if a pencil of rays is projective to a pencil of planes, the center of the former lying on the axis of the latter, two rays of the pencil may lie in the planes corresponding to them while others do not.

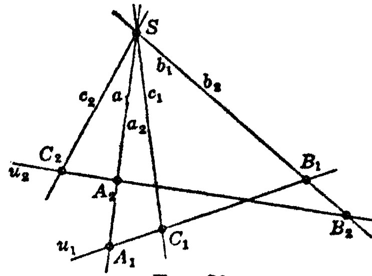


FIG. 29

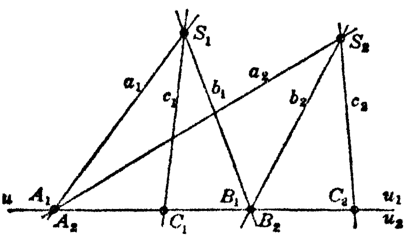


FIG. 30

54. In Superposed Projective Forms, Not Identical, There Are at Most Two Self-Corresponding Elements. Suppose there are given two superposed projective forms of the first order, two ranges of points, u_1 and u_2, for example, in which there are three self-corresponding elements, A_1 and A_2 coinciding at A, B_1 and B_2 coinciding at B, C_1, and C_2 coinciding at C (Fig. 31); and let us assume that in the segment AB in which C does not lie there is a point P_1 of the range u_1 which does not coincide with its corresponding point P_2 of u_2. Then if P_1 moves towards A, P_2 will also

move towards A (§ 51), and the two corresponding points will coincide at A, though they may have coincided earlier, and for the first time, say, at a point M. Or, if P_1 moves towards B, P_2 will also move towards B, at which point the two will coincide, though they may have coincided earlier, and for the first time, say, at a point N. On the assumption, therefore, that a point P_1 of the segment AB does not coincide with its corresponding point P_2, a segment MN has been determined, coinciding with or a part of AB, in which no point coincides with its corresponding point excepting only the end-points.

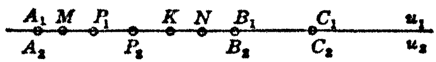


FIG. 31

If, now, K is the harmonic conjugate of C relative to M and N, it falls in the segment MN which does not contain C, that is, in the segment MN in which there is no self-corresponding point.

But the two ranges of points u_1 and u_2 are projective and a harmonic set of points $M_1K_1N_1C_1$ in u_1 must correspond to a harmonic set $M_2K_2N_2C_2$ in u_2; and since in these two sets three pairs of homologous elements M_1 and M_2, N_1 and N_2, C_1 and C_2 coincide, the fourth pair K_1 and K_2 must also coincide, and there is at least one point of the segment MN which is self-corresponding, a conclusion contrary to that which follows from the assumption that P_1 and P_2 do not coincide.

There is therefore no point in the segment AB which is not self-corresponding and the same is true of any other segment of the line. That is to say, all pairs of homologous points in the superposed ranges coincide.

It follows by projection, or it may be proved similarly, that if two projective pencils of rays or pencils of planes are

superposed and three pairs of homologous elements in the two pencils coincide, then all pairs of homologous elements coincide. Hence we may state the following theorem.

THEOREM. *In two superposed projective forms of the first order, if three elements of the one form coincide with the homologous elements of the other, then all pairs of homologous elements in the two forms coincide.*

Two pencils of rays are related projectively if to each ray of one pencil is correlated the ray of the other parallel to it or at right angles to it, and consequently it follows from the theorem just stated that in two projectively related pencils of rays lying in the same plane, if three rays of one pencil are parallel to, or are at right angles to, the homologous rays in the other, then all pairs of homologous rays in the two pencils are parallel, or are at right angles.

55. Projective Forms in Perspective. From the theorem of § 54 the following two reciprocal theorems may be deduced.

THEOREM. *If two projectively related pencils of rays lying in the same plane are so situated that three points of intersection of pairs of homologous rays lie on one straight line, then all points of intersection of pairs of homologous rays will lie on that straight line and the common ray of the two pencils is self-corresponding.*

For, on this line there will lie two projectively related ranges of points, sections of the two pencils of rays, having three self-corresponding elements.

THEOREM. *If two projectively related ranges of points lying in the same plane are so situated that three of the lines joining pairs of homologous points pass through one point, then all lines joining pairs of homologous points will pass through that point and the common point of the two ranges is self-corresponding.*

For, with this point as center there will lie two projectively related pencils of rays, projections of the two ranges of points, having three self-cor-

Hence all elements are self-corresponding and a pair of corresponding rays will intersect at each point of the line.

The two pencils of rays are therefore projections of the same range of points and are perspective to the range of points and to each other. Hence the common ray of the two pencils is self-corresponding (§ 48).

Conversely:

THEOREM. *If two projectively related pencils of rays lying in the same plane and not concentric are so situated that their common ray is self-corresponding, the two pencils are perspective and all pairs of homologous rays intersect in points of one straight line.*

For, the line joining the points of intersection of any two pairs of homologous rays will cut the two projective pencils in superposed projective ranges of points having three self-corresponding points; namely, the two points determining the line and the point in which the line cuts the common ray of the two pencils. Hence, all points in these two

responding elements. Hence all elements are self-corresponding and each ray of the superposed pencils will join a pair of homologous points of the ranges.

The two ranges of points are therefore sections of the same pencil of rays and are perspective to the pencil of rays and to each other. Hence the common point of the two ranges is self-corresponding (§ 48).

Conversely:

THEOREM. *If two projectively related ranges of points lying in the same plane and not coincident are so situated that their common point is self-corresponding, the two ranges are perspective and all lines joining pairs of homologous points pass through one point.*

For, the two projectively related ranges of points are projected from the point of intersection of the rays joining any two pairs of homologous points by two superposed projective pencils of rays having three self-corresponding rays; namely, the two rays determining the point of projection and the ray joining this point to the common point of the two

ranges are self-corresponding and all pairs of homologous rays in the two pencils intersect on this straight line.

ranges. Hence, all rays of the two pencils are self-corresponding and join pairs of homologous points in the two ranges.

By projecting the plane figures of these two converse theorems from a point outside the plane, we obtain the following theorems.

THEOREM. *If two projectively related pencils of planes whose axes intersect are so situated that their common plane is self-corresponding, the lines of intersection of pairs of homologous planes all lie in one plane and pass through one point, forming a pencil of rays whose center is the common point of the axes of the two pencils of planes.*

THEOREM. *If two projectively related pencils of rays which are concentric but not in the same plane, are so situated that their common ray is self-corresponding, the planes determined by pairs of homologous rays all intersect in one line, forming a pencil of planes whose axis passes through the common center of the two pencils of rays.*

It will be noted that these last two theorems may also be derived from the preceding converse theorems by reciprocation but in reverse order; that is, the space theorem on the left is the dual of the plane theorem on the right and *vice versa.*

56. To Correlate Two Given Ranges of Points Projectively. Suppose there are given in a plane two ranges of points, u_1 and u_2, not lying on the same straight line, and it is required to so correlate them, point to point, that they will be projectively related; that is, to correlate them so that any set of harmonic elements in the one will correspond to a set of harmonic elements in the other.

Since three points of a harmonic set determine the fourth point when it is known to which of the three the fourth is to

be conjugate, it is clear that not more than three points in one of the given ranges and their corresponding points in the other may be selected at random.

In the given range u_1, let the points A_1, B_1, C_1 correspond, respectively, to the points A_2, B_2, C_2 of the range u_2 (Fig. 32), these points being chosen at random. On the line A_1A_2 through a pair of corresponding points, select two points S_1 and S_2 and from them project the ranges u_1 and u_2, respectively.

Since the ranges u_1 and u_2 are to be so correlated projectively that the points A_1, B_1, C_1, will correspond to the points A_2, B_2, C_2, respectively, the pencils of rays S_1 and S_2 will be so correlated projectively that the rays $S_1(A_1, B_1, C_1)$ are homologous to the rays $S_2(A_2, B_2, C_2)$. In these two projective pencils the common ray S_1S_2 is self-corresponding. The pencils are therefore perspectively related (§ 55) and all pairs of corresponding rays in them will intersect in points of a straight line.

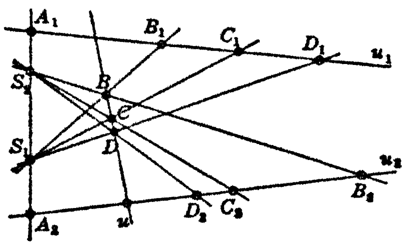


FIG. 32

The intersections, B and C, of the homologous rays S_1B_1 and S_2B_2, S_1C_1 and S_2C_2, determine a line u, and on this line all pairs of homologous rays of the pencils S_1 and S_2 must intersect. Hence to any point D_1 of u_1 the corresponding point D_2 of u_2 may be determined by drawing S_1D_1 intersecting the line u at D, and drawing S_2D intersecting u_2 in the required point D_2.

To every point of u_1 there is thus determined a corresponding point in u_2 and the two ranges are so correlated that to any set of harmonic points P_1, Q_1, R_1, S_1, of u_1 there

will correspond a set of harmonic points P_2, Q_2, R_2, S_2, in u_2. The two ranges u_1 and u_2 are therefore correlated projectively, the points A_1, B_1, C_1, and their homologous points A_2, B_2, C_2, having been chosen arbitrarily.

The question arises whether a different choice of centers S_1 and S_2 on the line A_1A_2, or if the use of the line B_1B_2 or C_1C_2 instead of A_1A_2, would have yielded a different point D_2, say D'_2, homologous to D_1, and so would have established a different correlation between the ranges u_1 and u_2.

In either case, the range of points $A_2B_2C_2D'_2 \cdots$ on the line u_2 would be projective to the range $A_1B_1C_1D_1 \cdots$ on the line u_1, which by the construction of Fig. 32 is projective to the range $A_2B_2C_2D_2 \cdots$. Consequently, the range $A_2B_2C_2D'_2 \cdots$ would be projective to the superposed range $A_2B_2C_2D_2 \cdots$ and these ranges have three self-corresponding points. Therefore, D and D' must coincide and the correlation established by the construction is unique.

THEOREM. *A projective relation is established uniquely between two ranges of points when three points in one range and their homologous points in the other have been selected.*

If the two ranges of points do not lie in the same plane, either of them may be projected from any center on a plane passing through the other, and the problem to establish a projectivity between them reduces to that already solved. If the two ranges lie on the same line, one of them may be projected from a chosen center on a different line and the problem again is reduced to the one here solved.

57. To Correlate Two Given Pencils of Rays Projectively. Suppose there are given in a plane two pencils of rays S_1 and S_2 which are not concentric and it is required to so correlate them that they will be projectively related; that is, so that any set of harmonic rays in the one pencil will correspond to a set of harmonic rays in the other. This is clearly the reciprocal problem to that solved in § 56.

As in the solution of that problem, three rays a_1, b_1, c_1, of the given pencil S_1 may be correlated, respectively, to three rays a_2, b_2, c_2, of the pencil S_2, these pairs of homologous rays being chosen arbitrarily.

Let one pair of homologous rays, a_1 and a_2, intersect at A (Fig. 33), and through A draw two lines u_1 and u_2, the first of which cuts the rays a_1, b_1, c_1, at the points A_1, B_1, C_1, and the second cuts the rays a_2, b_2, c_2, at the points A_2, B_2, C_2, respectively.

Since the given pencils of rays S_1 and S_2 are to be so correlated projectively that the rays a_1, b_1, c_1, are homologous to the rays a_2, b_2, c_2, respectively, the two ranges of points u_1 and u_2 will be projective, A_1, A_2; B_1, B_2; C_1, C_2, being pairs of homologous points. But the common point of these two ranges is self-corresponding. Hence the lines joining pairs of homologous points will pass through one point (§ 55). If then B_1B_2 and C_1C_2 intersect at S, the line joining any other pair of homologous points in u_1 and u_2 will pass through S. Let d_1 be any other ray of the pencil S_1 cutting u_1 at D_1. If the line D_1S is drawn cutting u_2 in D_2, the ray S_2D_2, or d_2, will be homologous to d_1.

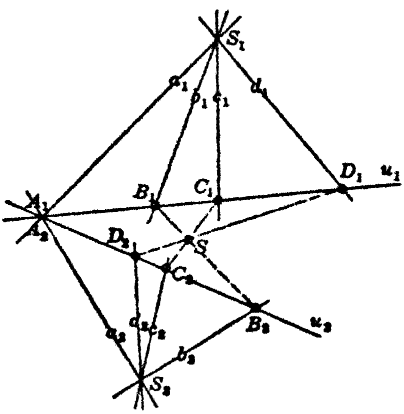


Fig. 33

By this means the pencils of rays S_1 and S_2 are so correlated that to any set of harmonic rays p_1, q_1, r_1, s_1, in S_1, there will correspond a set of harmonic rays p_2, q_2, r_2, s_2, in S_2, and the two pencils are therefore correlated projectively.

58. Projective Forms, the First and Last of a Series in Perspective. In the solution of the two preceding reciprocal problems (§§ 56, 57) a third form has been found in each case which is perspective to both of the given forms, and the two projectively related forms are made to appear as the first and last of a series of forms in perspective. Moreover, any two primitive forms of the first order may be correlated projectively by establishing a projective relation between projections or sections of these forms. Hence we have the following theorem.

THEOREM. *Any two one-dimensional primitive forms which are projectively related may be made to appear as the first and last of a series of forms in perspective and, conversely, if two one-dimensional primitive forms are the first and last of a series of forms in perspective, they are projectively related.*

This property justifies the use of the term "projective" in the definition of § 50 and has itself frequently been taken as the definition of projectively related forms.[1]

59. Metric Properties of Projective Forms. It was shown at the close of Chapter IV (§ 43) that the cross-ratio of four points on a line or of four lines through a point is unaltered by projection.

In two projectively related primitive forms of the first order, since we may pass from any four elements of the one to the corresponding elements of the other by a series of projections (§ 58), we may state the following theorem.

THEOREM. *The cross-ratio of any four elements in one of two projectively related primitive forms of the first order is equal to the similar cross-ratio of the corresponding elements in the other.*

On the other hand, if we have given two primitive forms

[1] Poncelet (1788–1867) used this property as the definition of projectivity. It was adopted also by Cremona (1830–1903); see his *Projective Geometry*, Oxford, 1885, § 40.

of the first order—two ranges of points, u_1 and u_2, in the same plane, for example—so correlated that the cross-ratio of four points P_1, Q_1, R_1, S_1, chosen at random in u_1, is equal to the similar cross-ratio of the corresponding points P_2, Q_2, R_2, S_2, of u_2, these two forms are projectively related.

Suppose it is given that in the two ranges of points,

$$\frac{P_1Q_1 \cdot R_1S_1}{P_1R_1 \cdot Q_1S_1} = \frac{P_2Q_2 \cdot R_2S_2}{P_2R_2 \cdot Q_2S_2}.$$

Then, if the points P_1, Q_1, R_1, S_1, are projected from P_2, and P_2, Q_2, R_2, S_2 are projected from P_1 (Fig. 34), the rays P_1Q_2 and P_2Q_1 intersecting at K, and the rays P_1R_2 and P_2R_1

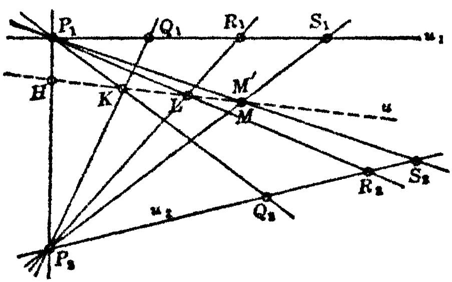


FIG. 34

intersecting at L, the rays P_1S_2 and P_2S_1 must intersect in a point of the line KL.

For, if otherwise, suppose P_1S_2 cuts KL at M while P_2S_1 cuts it at M'. Then, if KL cuts P_1P_2 at H, we have, by § 43,

$$\frac{P_1Q_1 \cdot R_1S_1}{P_1R_1 \cdot Q_1S_1} = \frac{HK \cdot LM'}{HL \cdot KM'},$$

and also

$$\frac{P_2Q_2 \cdot R_2S_2}{P_2R_2 \cdot Q_2S_2} = \frac{HK \cdot LM}{HL \cdot KM}.$$

But it was assumed that the left-hand members of these

two equations are equal. Therefore

$$\frac{HK \cdot LM'}{HL \cdot KM'} = \frac{HK \cdot LM}{HL \cdot KM},$$

or

$$\frac{LM'}{M'K} = \frac{LM}{MK},$$

in which case M and M' must coincide.

That is to say, if the cross-ratio of four points of u_1, chosen at random, equals the similar cross-ratio of the corresponding points of u_2, we may pass from the points of one range to the corresponding points of the other by a series of perspective forms and the two ranges are therefore projectively related.

More generally, we may state the following theorem.

THEOREM. *If two one-dimensional primitive forms are so correlated that the cross-ratio of four elements of the one, chosen at random, equals the similar cross-ratio of the corresponding elements of the other, the two forms are related projectively.*

This property also has been taken as a definition of projectivity.[1]

60. Forms Projective after Permutation. In § 44 it was shown that the cross-ratio of four points on a line is equal to the similar cross-ratio of the same four points taken in any order, provided that when two points are interchanged, the other two are also interchanged. From this we have the following theorem.

THEOREM. *Four elements of a one-dimensional primitive form are projective to any permutation of those elements in which two of them, and also the other two, are interchanged.*

For example, the points P, Q, R, S, of a straight line are projective to the same points in the order Q, P, S, R, or in the order R, S, P, Q.

[1] Steiner (1796–1863), *Systematische Entwickelung*, 1832. Also Chasles (1793–1880), *Géométrie Supérieure*, 1880.

EXERCISES

1. In two projectively related pencils of rays lying in the same plane and not concentric, there are at most two pairs of homologous rays parallel, unless all pairs are parallel.

2. In two correlated pencils of rays lying in the same plane, if pairs of homologous rays make a constant angle with each other, or if they make equal angles with a fixed line, the two pencils are projectively related.

3. In solving the problem, to relate two given ranges of points projectively (§ 56), what would be the result if the line u should pass through the common point of u_1 and u_2? What is the corresponding relation in the solution of the reciprocal problem (§ 57)?

4. Given two pencils of rays with finite centers, perspectively related, find two rays at right angles in one of them which correspond, respectively, to two rays at right angles in the other. Hence, show that in two projectively related pencils of rays with finite centers, there is always a pair of orthogonal rays in one, which correspond, respectively, to a pair of orthogonal rays in the other.

5. Given four points, A, B, C, D, on a straight line, show by repeated projections that $ABCD$ is projective to $BADC$, and also to the same four points in any order in which two of them and the other two are interchanged; that is, prove the theorem of § 60 by means of a diagram.

SUGGESTION. Project A, B, C, D, from any point S into the collinear points A, E, F, G and project these from D into B, E, K, S; and these again from F into B, A, D, C. Similarly for other permutations.

6. Given two fixed straight lines, u_1 and u_2, intersecting in O, and two fixed points, S_1 and S_2, collinear with O. A straight line rotates about a fixed point V and intersects u_1 and u_2 at A_1 and A_2, respectively. Show that the locus of the point of intersection of S_1A_1 and S_2A_2 is a straight line. (Chasles, *Géométrie Supérieure*, 1880, § 342; also Pappus, *Mathematicae Collectiones* VII.)

SUGGESTION. The ranges of points described by A_1 and A_2 are perspective and the common ray of the pencils S_1 and S_2 is self-corresponding.

7. State and prove the theorem reciprocal to that of Exercise 6.

8. Given three collinear points, P, S_1, S_2, and two fixed straight lines, u_1 and u_2, intersecting at O. Through P, there is drawn a straight line

p intersecting u_1 and u_2 at A_1 and A_2, respectively. If the straight lines S_1A_1 and S_2A_2 intersect at M, show that as p rotates about P, the locus of M is a straight line passing through O. (Chasles, *Géométrie Supérieure*, § 343.)

This exercise may be stated in the following form in which it is the equivalent of Desargues' theorem on perspective triangles (§ 30).

THEOREM. If the three sides of a variable triangle MA_1A_2 rotate about three fixed collinear points, P, S_1, S_2, while two vertices, A_1 and A_2, move along fixed straight lines intersecting at O, the third vertex will also move along a straight line passing through O.

9. State and prove the theorem reciprocal to that of Exercise 8.

10. If the four vertices, A, B, C, D, of a variable complete quadrangle move, respectively, on four fixed straight lines passing through one point O, while three of the sides, AB, BC, CD, rotate about three fixed points, the remaining three sides will also rotate about fixed points, and these six fixed points form the vertices of a complete quadrilateral; that is, they lie by threes on four fixed straight lines. (Cremona, *Projective Geometry*, Oxford, 1885, § 111.)

11. Given three rays, a, b, c, through a point, making fixed angles with each other, and three fixed points, A, B, C, on a straight line. Place the two forms in such a position that the ray a will pass through A, b will pass through B, and c through C. Hence, place a pencil of rays and a range of points projective to it, in perspective position.

SUGGESTION. Use Euclid, Book III, Prop. 33.

12. If a pencil of planes u is perspective to a pencil of rays S, the axis of the pencil of planes is normal to one of the two rays at right angles in S which correspond to two planes at right angles in u.

Hence, each of the following problems admits of two solutions: Given a pencil of rays S and a pencil of planes u projective to it; (1) to pass a plane through a given point which shall intersect u in a pencil of rays congruent to S; and (2) to find an axis from which S is projected by a pencil of planes congruent to u.

13. Place a pencil of rays S and a pencil of planes u in such relative positions that three given rays a, b, c, of S shall lie in three given planes α, β, γ, of u.

14. Construct a plane cutting the faces α, β, γ, of a triangular prism in a triangle equiangular with a given triangle.

CHAPTER VI

CURVES AND PENCILS OF RAYS OF THE SECOND ORDER

61. Forms of the Second Order in a Plane. If two projectively related pencils of rays lie in the same plane and are neither concentric nor perspective, at most two pairs of homologous rays intersect in points of any straight line (§ 54). If three pairs should intersect in points of a straight line, the two pencils would be perspective, and all pairs of homologous rays would intersect on that line. The pencils would likewise be perspective if their common ray were self-corresponding.

Similarly, if two projectively related ranges of points lie in one plane and are neither collinear nor perspective, at most two rays joining pairs of homologous points can pass through one point (§ 54).

Moreover, a sequence of elements in one of two projectively related forms corresponds to a similar sequence of elements in the other (§ 51). Hence we have the following reciprocal theorems.

THEOREM. *If two projectively related pencils of rays lie in one plane and are neither concentric nor perspective, the points of intersection of pairs of homologous rays form a sequence of points of which not more than two lie on any straight line.*

The form so ***generated*** is called ***a point-curve or a range of points of the second order.***

THEOREM. *If two projectively related ranges of points lie in one plane and are neither collinear nor perspective, the rays joining pairs of homologous points form a sequence of rays of which not more than two pass through any point.*

The form so ***generated*** is called ***a line-curve or a pencil of rays of the second order.***

If the generating pencils of rays are perspective, that is, if their common ray is self-corresponding, the resulting point-curve or range of points is of the first order, and all points of it lie on one straight line (§ 55).

If the generating ranges of points are perspective, that is, if their common point is self-corresponding, the resulting line-curve or pencil of rays is of the first order, and all rays of the pencil pass through one point (§ 55).

It is by reason of the property that two points of the curve generated by two projective pencils of rays, and not more than two, may lie on one straight line, and that two rays of the pencil generated by two projective ranges of points, and not more than two, may pass through one point, that the generated forms are said to be of the second order. Of the points of a curve of the first order, not more than one may lie on any straight line, and of the rays of a pencil of the first order, not more than one may pass through any point, except that in the first case all points lie on a particular straight line and in the second case all rays pass through a particular point.

DEFINITION. A point-curve of the second order is a sequence or range of points of which two and not more than two may lie on one straight line. Such a curve is generated by two projective pencils of rays which lie in one plane and are neither perspective nor concentric.

DEFINITION. A line-curve or pencil of rays of the second order [1] is a sequence or system of rays of which two and not more than two may pass through one point. Such a pencil of rays is generated by two projective ranges of points which lie in one plane and are neither perspective nor collinear.

[1] Whenever the term "pencil of rays" is used, it will be understood to refer to the primitive form of the first order designated by that name unless it is otherwise stated. Also the term "curve" will refer to the point-curve unless line-curve is specified.

As a matter of interest it may be stated here, leaving the demonstration till later, that the point-curve of the second order is identical with the conic section of ancient geometry and the line-curve or pencil of rays of the second order is identical with the system of tangents to a conic.

62. Forms of the Second Order in Space. In three-dimensional geometry, we have the following theorems corresponding to those of § 61.

THEOREM. *Two concentric pencils of rays which are projectively related and do not lie in the same plane generate, in general, a* ***pencil of planes of the second order.*** *The planes of this pencil all pass through the common center of the generating pencils but not more than two of them intersect in any line.*

The common center of the generating pencils of rays is the ***vertex*** of the pencil of planes.

A section of this form not passing through the common center consists of two projectively related ranges of points and the lines joining homologous pairs. It is, therefore, a pencil of rays of the second order.

If the two generating pencils of rays are perspective, that is, if their common ray is self-corresponding, the planes determined by pairs of homologous

THEOREM. *Two pencils of planes whose axes intersect and which are projectively related generate, in general,* ***a cone of the second order.*** *The rays of this cone all pass through the common point of the axes of the generating pencils but not more than two of them lie in any plane.*

The common point of the axes of the generating pencils of planes is the ***vertex*** of the cone.

A section of this form not passing through the common point of the axes consists of two projectively related pencils of rays and the points of intersection of homologous pairs. It is, therefore, a curve of the second order.

If the two generating pencils of planes are perspective, that is, if their common plane is self-corresponding, the rays determined by pairs of homol-

rays all intersect in one line and the form generated is a pencil of planes of the first order.

ogous planes all lie in one plane and the form generated is a pencil of rays of the first order.

The forms of the second order defined in the preceding reciprocal theorems, namely, the pencil of planes of the second order and the cone of the second order, may be derived from the corresponding forms in the plane by projecting the latter from any center outside their plane. Thus,

A point-curve of the second order and its generating pencils of rays is projected from any point not lying in its plane, by a cone of the second order and its generating pencils of planes.

Reciprocally,

Any section of a cone of the second order by a plane not passing through its vertex is a point-curve of the second order.

A line-curve of the second order with its generating ranges of points is projected from any point not lying in its plane, by a pencil of planes of the second order and its generating pencils of rays.

Reciprocally,

Any section of a pencil of planes of the second order by a plane not passing through its vertex is a line-curve of the second order.

When the vertex of a cone of the second order is infinitely distant, its rays are parallel and the form is called a ***cylinder*** of the second order.

63. The Centers of Generating Pencils and the Bases of Generating Ranges of Points Are Elements of the Forms Generated. If S_1 and S_2 are two pencils of rays of the first order lying in the same plane, projective but not perspective, the ray S_1S_2 of S_1 (Fig. 35) intersects its corresponding ray of S_2 at the point S_2. Hence S_2 is a point of the curve of the second order generated by S_1 and S_2. Similarly, S_1 is a

point of this curve since the ray S_2S_1 of S_2 intersects its corresponding ray of S_1 at the point S_1.

Also, if u_1 and u_2 are two ranges of points of the first order lying in the same plane, projective but not perspective, (Fig. 36), the point of u_1 common to u_1 and u_2 corresponds to some point of u_2, so that the ray u_2 itself passes through a

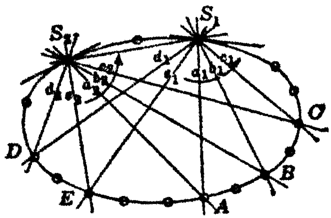


FIG. 35

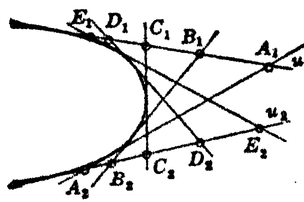


FIG. 36

pair of homologous points in the two given ranges and is a ray of the line-curve. Similarly, the ray u_1 is a ray of the line-curve since it also joins a pair of homologous points in the given ranges. Hence, we have the following theorems.

THEOREM. *The curve of the second order generated by two projective pencils of rays passes through the centers of these pencils; that is, the centers of the generating pencils are points of the curve.*

THEOREM. *The pencil of rays of the second order generated by two projective ranges of points contains the rays on which those ranges lie; that is, the bases of the generating ranges are rays of the pencil.*

From these reciprocal theorems, by projection or by duality, the following theorems are derived.

THEOREM. *The axes of two projective pencils of planes which generate a cone of the second order are rays of the cone.*

THEOREM. *The planes of two projective pencils of rays which generate a pencil of planes of the second order are planes of the pencil.*

64. Tangents and Points of Contact. A ray of the generating pencil S_1 (Fig. 37) cuts the curve of the second order, in general, in two points; namely, at S_1 and at the point in which it intersects the homologous ray of S_2. There is, however, one ray of S_1 which meets the curve only at S_1; namely, that ray of S_1 which is homologous to the ray S_2S_1 of the generating pencil S_2. For this ray, it may be said that the two points of intersection with the curve coincide, and this particular ray is called the ***tangent*** to the curve at S_1. In the same way there is a tangent to the curve at S_2; namely, the ray of S_2 homologous to the ray S_1S_2 of S_1.

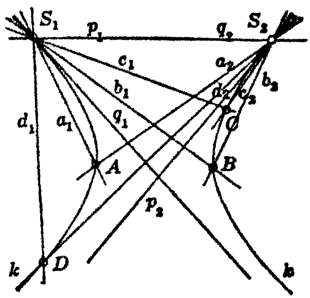


FIG. 37

Similarly, through any point K_1 of the range u_1 (Fig. 38), there will pass in general two rays of the pencil of the second order; namely, the ray u_1 and the ray joining K_1 to the homologous point of u_2. There is, however, one point of u_1 through which only one ray of the pencil will pass; namely, the point Q_1 which corresponds to the point Q_2 of u_2, common to u_1 and u_2. In other words, the two rays of the pencil of the second order passing through the point Q_1 coincide. This particular point is called the ***point of contact***

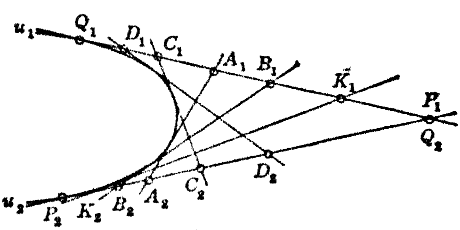


FIG. 38

in u_1. In the same way, there is a point of contact in u_2; namely, the point of u_2 homologous to the point P_1 common to u_2 and u_1.

From similar considerations, it may be seen that the cone of the second order which is generated by two projective pencils of planes whose axes intersect, has a tangent plane through the axis of each of the generating pencils; or, this same conclusion is reached if a curve of the second order is projected from any point outside its plane. Likewise, in a pencil of planes of the second order, there is a ray of contact in each of the planes in which the generating pencils lie.

65. Forms of the Second Order Are Determined by Five Elements. Since three pairs of homologous elements in two one-dimensional primitive forms determine a projective correspondence between them (§§ 56, 57), the following reciprocal statements may be made.

The centers of two generating pencils of rays, S_1 and S_2, and three additional points A, B, C, determine, in general, a curve of the second order passing through the five points.

For the given centers may be joined to each of the three points, and the lines so drawn to the same point may be designated as homologous rays in the two pencils. To any ray in the one pencil, there may then be determined the homologous ray in the other, and the point of intersection of these rays is a point of the curve.

The bases of two generating ranges of points, u_1 and u_2, and three additional rays, a, b, c, determine, in general, a pencil of rays of the second order of which the five rays are elements.

For the intersections of each of the three rays with the two given bases may be designated as homologous points in the two generating ranges. To any point in the one range may then be determined the homologous point in the other, and the ray joining these points is a ray of the pencil.

Since the projective relation thus established between the two pencils of rays S_1 and S_2 is unique (§ 57), there is but one curve of the second order passing through S_1, S_2, and the three other given points.

It is assumed that the three points A, B, C, do not lie on one line, otherwise the form determined by the five points would consist of the range of points on the line A, B, C, and the line S_1S_2, every point of which is common to two homologous rays of the generating pencils; neither do two of the points A, B, C, lie on a line with S_1 or S_2.

Since the projective relation thus established between the two ranges of points u_1 and u_2 is unique (§ 56), there is but one pencil of rays of the second order containing u_1, u_2, and the three other given rays.

It is assumed that the three rays a, b, c, do not pass through one point, otherwise the form determined by the five rays would consist of the pencil of rays of the first order whose center is the point of intersection of a, b, c, and the pencil whose center is the intersection of u_1 and u_2, every ray of which passes through a pair of homologous points of the generating ranges; neither can two of the rays a, b, c, intersect u_1 or u_2 at the same point.

In the curve of the second order generated by two projective pencils of rays S_1 and S_2, the tangent at S_1 corresponds to the ray S_2S_1 of S_2 (§ 64), and the tangent at S_2 corresponds to the ray S_1S_2 of S_1. Hence the projectivity of the two generating pencils of rays S_1 and S_2 will be determined, and consequently the curve itself will be determined, if there are given, besides the two centers, two points of the curve and the tangent at S_1, or at S_2; or one point of the curve and the tangents at both S_1 and S_2, For, in either case, there are given three rays through one center and their homologous rays through the other.

Similarly, a pencil of rays of the second order is determined when there are given, besides the two bases u_1 and u_2,

two other rays and the point of contact in u_1 or u_2, or one other ray and the points of contact in both u_1 and u_2.

66. The Centers of the Generating Pencils of Rays and the Bases of the Generating Ranges of Points Are Not Different from Other Elements of the Forms Generated. Thus far the centers of the pencils of rays generating a curve of the second order may well be considered special points of the curve since they play a distinctive part in the construction; and the same is true of the bases of the generating ranges of points in relation to the pencil of rays of the second order.

It remains to show that any other points of the curve might as well be chosen for centers of the generating pencils, and that any other rays of the pencil of the second order might as well be chosen for bases of the generating ranges of points. With this it will be shown that these elements in the two forms are not different from other elements.

In Fig. 39, in which S_1 and S_2 are the centers of the pencils of rays generating a point-curve of the second order (§ 57), A, B, C, and P are any points whatsoever of the curve and the lines u_1 and u_2 are drawn at random through the point A. If S_1S is drawn cutting u_1 at L_1 and u_2 at L_2, the intersection L_2 is a point of the curve. For the ray S_1L_1 of the pencil S_1 intersects its homologous ray S_2L_2 of S_2 at L_2. Similarly, if S_2S intersects u_1 at M_1, then M_1 is a point of the curve. Moreover, since u_1 and u_2 are drawn through the point A at random, M_1 and L_2 are any points whatsoever of the curve determined by S_1, S_2, A, B, and C.

Suppose, now, the points S_1, S_2, P, L_2, and M_1 of the curve remain fixed, so that the curve itself remains fixed (§ 65) while the point A moves along the curve. The center S will remain fixed since it is the intersection of the fixed lines S_1L_2 and S_2M_1. The rays M_1A and L_2A will rotate about M_1 and L_2, respectively, and the ray D_1D_2 will describe a

pencil of rays about S of which the lines S_1P and S_2P are sections. The ranges of points marked out on these lines by D_1 and D_2 are consequently perspective.

The pencils of rays described by M_1A and L_2A are perspective to the ranges S_1P and S_2P and are therefore projective to each other. That is to say, while the point A describes the curve generated by the projective pencils of rays whose centers are S_1 and S_2, it is also the point of

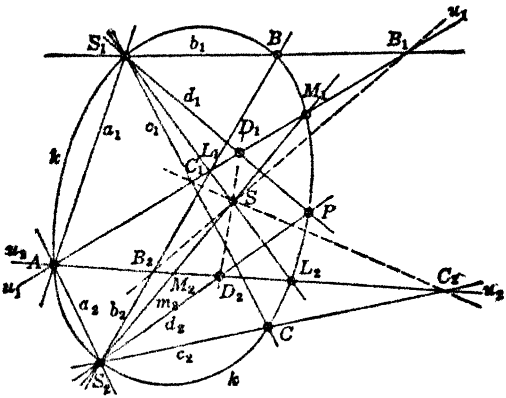


FIG. 39

intersection of homologous rays in the projective pencils whose centers are M_1 and L_2, these being any other points whatsoever of the curve. Hence it follows that the curve of the second order generated by two projective pencils of rays S_1 and S_2 is generated likewise by two projective pencils of rays whose centers are any other points of the curve, and consequently, the centers S_1 and S_2 are not different from other points of the curve. This property may be stated in the following theorem.

THEOREM. *A curve of the second order is projected from any two of its points by projectively related pencils of rays.*

Similarly, in Fig. 40 illustrating the process of finding additional rays of the pencil of the second order of which the rays u_1, u_2, A_1A_2, B_1B_2, C_1C_2 are given, the ray D_1D_2 is any ray whatsoever of the pencil (§ 56). If the line of perspectivity u intersects u_1 at R_1 and u_2 at Q_2, the rays S_1Q_2 and S_2R_1, or Q_1Q_2 and R_1R_2, are likewise any rays whatsoever of the pencil since S_1 and S_2 are chosen at random on A_1A_2.

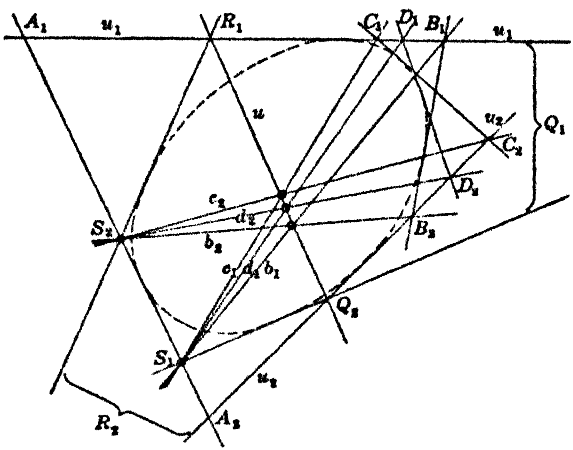


Fig. 40

Suppose, now, the figure remains fixed except that the ray A_1A_2 takes up successive positions in the pencil of the second order; that is, A_1A_2 joins successively pairs of homologous points in the projective ranges u_1 and u_2. The rays Q_1Q_2 and R_1R_2 remain fixed and on them the moving ray A_1A_2 will mark out ranges of points which are projected from the fixed points D_1 and D_2, by perspective pencils of rays, homologous rays in these pencils intersecting on the fixed line u. These ranges of points on Q_1Q_2 and R_1R_2 are therefore projective, and the ray A_1A_2 not only joins pairs of homologous points in the projective ranges u_1 and u_2, but

it also joins pairs of homologous points in the projective ranges Q_1Q_2 and R_1R_2, these being any two rays whatsoever of the pencil. In the pencil of rays of the second order, therefore, u_1 and u_2 are not different from other rays. From this the following theorem may be stated.

THEOREM. *Any two rays of a pencil of the second order are cut by the other rays of the pencil in projective ranges of points.*

67. Two Forms of the Second Order Are Identical if They Have Five Elements in Common. Since the centers of the projective pencils of rays generating a curve of the second order and the bases of the projective ranges of points generating a pencil of rays of the second order are not different from other elements of the forms generated, we have the following properties.

Any two points of a curve of the second order may be taken as the centers of the generating pencils of rays. Consequently, at every point of the curve there is a tangent.

Any two rays of a pencil of the second order may be taken as bases of the generating ranges of points. Consequently, on every ray of the pencil there is a point of contact.

Moreover, five elements of a curve of the second order or of a pencil of rays of the second order are sufficient to determine the curve or the pencil uniquely (§ 65) and consequently the following statements are true.

Two curves of the second order are identical if they have in common five points, or four points and the tangent at one of them, or three points and the tangents at two of them.

Two pencils of rays of the second order are identical if they have in common five rays, or four rays and the point of contact in one of them, or three rays and the points of contact in two of them.

Similarly, two cones of the second order are identical if they have in common five rays, or four rays and the tangent plane in one of them, or three rays and the tangent planes in two of them.

Similarly, two pencils of planes of the second order are identical if they have in common five planes, or four planes and the ray of contact in one of them, or three planes and the rays of contact in two of them.

68. Pascal's Theorem. In Fig. 39 let us consider the hexagon $S_1PS_2M_1AL_2$ whose vertices are any six points whatsoever of a curve of the second order (§ 66).

The pairs of opposite sides of this hexagon, S_1P and M_1A, PS_2 and AL_2, S_2M_1 and L_2S_1, intersect in the points D_1, D_2, and S, respectively, and from the construction of the curve (§ 57) these three points necessarily lie on one straight line. From this we have the following important theorem.

THEOREM. *In any simple hexagon inscribed in a curve of the second order, the three pairs of opposite sides intersect in points of one straight line.*

This theorem as applied to the conic section of ancient geometry was discovered by Pascal in 1639, and was published the following year in his *Essai sur les Coniques.* It is commonly known as ***Pascal's Theorem.***

The converse of Pascal's theorem is equally true and may be stated in the following form.

THEOREM. *If a simple hexagon is such that the points of intersection of its three pairs of opposite sides lie on one straight line, a curve of the second order will pass through its six vertices.*

For, if the curve of the second order determined by any five of the vertices, S_1, S_2, M_1, A, L_2 (Fig. 39), should not pass through the sixth point P, but should cut the line S_1P at some other point P', then the line S_2P' would cut AL_2 at a point D_2' which could not lie on the line D_1S.

69. Construction of a Curve of the Second Order by Use of Pascal's Theorem. Pascal's theorem may be used to construct any required number of points of a curve of the second order of which five points are given. This problem may be stated as follows.

PROBLEM. *Given five points of a curve of the second order and an arbitrary line through one of them, to find where the line cuts the curve a second time.*

If A, B, C, D, E, are five given points of the required curve (Fig. 41) and p is an arbitrary line through A, the problem is to construct an inscribed hexagon having these five points for vertices and the line p for one side.

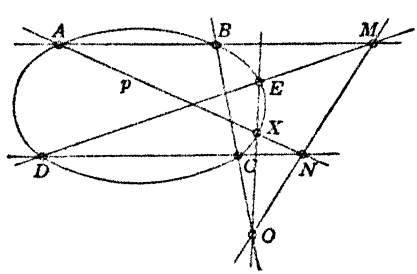


FIG. 41

The pairs of opposite sides of the hexagon may be taken in the order AB and DE, BC and EX, CD and XA, where X is a point of the curve on the line p, as yet unknown.

If AB and DE intersect at M, and CD cuts p at N, by Pascal's theorem, the third pair of opposite sides, BC and EX, must intersect in a point of the line MN. Suppose BC cuts MN at O, then EO can be drawn determining the required point X on p, the point where the line p cuts the curve a second time. If the point X should coincide with A, the line p would meet the curve only at A and would therefore be a tangent.

In this and similar constructions, the line on which the three pairs of opposite sides of the inscribed hexagon intersect is called the ***Pascal line*** for that hexagon. If the given points are taken in a different order, the Pascal line will be different but the same point X will be determined.

70. Brianchon's Theorem. Turning now to Fig. 40, consider the hexagon $S_1S_2R_1D_1D_2Q_2$ whose sides are any six rays of a pencil of the second order (§ 66).

The pairs of opposite vertices in this hexagon are S_1 and D_1, S_2 and D_2, R_1 and Q_2. From the construction of the pencil of rays of the second order (§ 56) the lines joining S_1 and D_1, S_2 and D_2, must intersect on the line joining the points R_1 and Q_2, and from this we have the following important theorem.

THEOREM. *In any simple hexagon whose sides are rays of a pencil of the second order, the three lines joining pairs of opposite vertices pass through one point.*

This theorem applied to the tangents to a conic section was discovered by Brianchon in 1806 and is known as ***Brianchon's Theorem***. It was first published in the *Journal de l'Ecole Polytechnique*, Volume XIII. The converse of Brianchon's theorem may be stated as follows.

THEOREM. *If a simple hexagon is such that its three principal diagonals pass through one point, its sides belong to a pencil of rays of the second order.*

71. Construction of a Pencil of Rays of the Second Order by Use of Brianchon's Theorem. Just as Pascal's theorem may be used to construct a curve of the second order from five given elements, so Brianchon's theorem lends itself to the construction of a pencil of rays of the second order. This problem takes the following form.

PROBLEM. *Given five rays of a pencil of the second order and an arbitrary point on one of them, to find the second ray of the pencil passing through that point.*

If a, b, c, d, e are the given rays of the pencil (Fig. 42), and P the given point on the ray a, the five rays may be taken as successive sides of a simple hexagon of which P is a vertex. In that case, the vertex (ab) is opposite to the vertex (de), and the vertex P is opposite to the vertex (cd).

If the lines joining these pairs of opposite vertices intersect at O, the line joining the vertex (bc) and the point O will determine on e the sixth vertex, and this vertex joined to P gives the required sixth ray of the pencil.

The point in which the three diagonals of the hexagon intersect is called the *Brianchon point* for that hexagon.

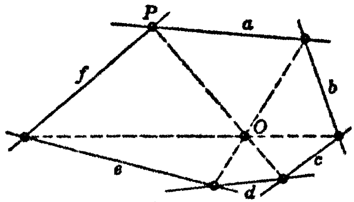


Fig. 42

72. Classification of Curves of the Second Order. In § 54, it was pointed out that if two projectively related pencils of rays or ranges of points of the first order are superposed, there are at most two pairs of homologous elements in the two forms which coincide, the two forms not being identical. That there may be two pairs of such self-corresponding elements follows from the fact that in correlating two forms projectively, three pairs of homologous elements may be selected at random (§§ 56, 57). Two of these pairs may be made to coincide, while if the third pair do not coincide, the forms are not identical.

In two superposed projective pencils of rays there may be no self-corresponding ray, as, for example, when pairs of homologous rays make a fixed angle with each other. Or, there may be one self-corresponding ray, or two such rays, and there are always two if the sequence of homologous rays is in opposite senses about the common center.

Similarly, in two superposed projective ranges of points, there may be no self-corresponding point, or there may be one such point, or at most two, and always two if the sequence of homologous points is in opposite senses along the line.

If the sequence of homologous elements in two projective

pencils of rays or ranges of points is in the same sense about the centers or along the base lines, the forms are said to be ***directly projective;*** if the sequence is in opposite senses, the forms are ***oppositely projective.***

It follows, therefore, that in two projective pencils of rays of the first order lying in the same plane and not concentric, there may be no pairs of homologous rays which are parallel, or there may be one pair, or there may be, at most, two pairs parallel unless all pairs are parallel.

A curve of the second order, therefore, may have no points infinitely distant, or it may have one point infinitely distant, or it may have two, and at most two, infinitely distant points. In other words, the infinitely distant line of the plane may have no points in common with a curve of the second order, or it may have one point, or it may have two points, and not more than two.

DEFINITION. A curve of the second order which has no point in common with the infinitely distant line of its plane is called ***an ellipse.***

DEFINITION. A curve of the second order which has one point in common with the infinitely distant line and to which the infinitely distant line of the plane is tangent is called ***a parabola.***

DEFINITION. A curve of the second order which has two points in common with the infinitely distant line, or which is cut by the infinitely distant line in two points, is called ***a hyperbola.***

DEFINITION. A tangent to a curve at an infinitely distant point is called ***an asymptote.***

A hyperbola therefore has two asymptotes, a parabola has one; namely, the infinitely distant line itself, and an ellipse has none.

73. A Circle is a Curve of the Second Order. From a well-known property of a circle, namely, that equal angles at the center or at the circumference subtend equal arcs, it is readily seen that a circle may be generated by two projective pencils of rays of the first order and is, consequently, a curve of the second order.

If a circle is projected from any two of its points (Fig. 43), the lines so drawn form two correlated pencils of rays of the first order, those being homologous rays which project the same point of the circle. The pencils so formed are projective since the angle between any two rays of the one pencil is equal to the angle between the homologous rays in the other. No pairs of homologous rays in these pencils are parallel, hence a circle fulfills the conditions for an ellipse.

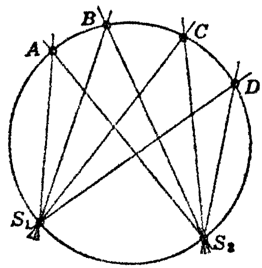


FIG. 43

74. Similarly Projective Ranges of Points. When two projectively related ranges of points u_1 and u_2 are such that the infinitely distant point of u_1 is homologous to the infinitely distant point of u_2, they are said to be ***similarly projective,*** and the pencil of rays of the second order generated by them will include the infinitely distant line of the plane among its elements.

If u_1 and u_2 are similarly projective ranges of points in the same plane, homologous segments in them are proportional. For two such projective ranges may be brought into perspective position by moving one of them, u_1 say, parallel to itself (Fig. 44) until their common point is self-corresponding. If u_1' is the new position of u_1, the infinitely distant point of u_1' is homologous to the infinitely distant point of u_2, since the translation of u_1 parallel to itself is equivalent to rotating it about its infinitely distant point. The lines

joining pairs of homologous points in u_1' and u_2 will then intersect in one point (§ 55) through which the infinitely distant line of the plane will pass, and the two ranges of points will appear as sections of a pencil of parallel rays.

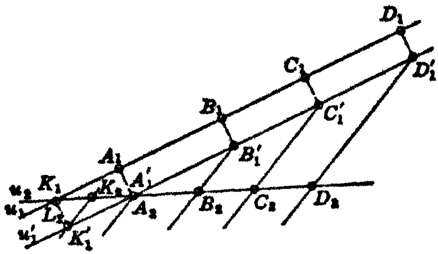


FIG. 44

In two such sections, homologous segments are proportional.

The same conclusion may be reached by considering the metric properties of projective ranges (§ 59). If u_1 and u_2 are correlated projectively by choosing the points A_1, B_1, C_1 of u_1 as homologous to the points A_2, B_2, C_2 of u_2 (§ 56), the points C_1 and C_2 being the infinitely distant points of the two ranges while D_1 and D_2 are any other pair of homologous points, from § 59 we have

$$\frac{A_1B_1 \cdot C_1D_1}{A_1C_1 \cdot B_1D_1} = \frac{A_2B_2 \cdot C_2D_2}{A_2C_2 \cdot B_2D_2},$$

or

$$\frac{A_1B_1}{B_1D_1} \cdot \frac{D_1C_1}{A_1C_1} = \frac{A_2B_2}{B_2D_2} \cdot \frac{D_2C_2}{A_2C_2}.$$

Hence

$$\frac{A_1B_1}{B_1D_1} \cdot \left(\frac{D_1A_1 + A_1C_1}{A_1C_1}\right) = \frac{A_2B_2}{B_2D_2} \cdot \left(\frac{D_2A_2 + A_2C_2}{A_2C_2}\right),$$

or

$$\frac{A_1B_1}{B_1D_1} \cdot \left(1 + \frac{D_1A_1}{A_1C_1}\right) = \frac{A_2B_2}{B_2D_2} \cdot \left(1 + \frac{D_2A_2}{A_2C_2}\right).$$

Since C_1 and C_2 are infinitely distant, the fractions D_1A_1/A_1C_1

and D_2A_2/A_2C_2 are indefinitely small, and may be disregarded. Hence

$$\frac{A_1B_1}{B_1D_1} = \frac{A_2B_2}{B_2D_2}.$$

In other words, homologous segments in the two ranges are proportional.

EXERCISES

1. Show that if five points of a plane are so situated that one of them lies within a simple quadrangle determined by the other four, the curve of the second order determined by the five points is a hyperbola.

2. If u_1 and u_2 are projective ranges of points lying in the same plane, show that it is always possible to bring two of their homologous points A_1 and A_2 into coincidence by rotating one of the ranges about some one of its points. In other words, show that two projective ranges of points may be brought into perspective position by rotating one of them about a fixed point.

3. If u_1 and u_2 are two fixed tangents to a circle whose center is O and v is a variable tangent intersecting u_1 and u_2 at A_1 and A_2, respectively, show that the angle A_1OA_2 is constant and hence that the tangent v determines projective ranges of points on u_1 and u_2. In other words, show that the system of tangents to a circle is a pencil of rays of the second order.

4. If the vertices of a variable triangle move on three fixed lines of its plane while two of the sides move parallel to themselves, the third side will move parallel to itself or will generate a pencil of rays of the second order of which the infinitely distant line of the plane is one ray.

5. A variable triangle A_1SA_2 moves in its plane so that the extremities of the base A_1 and A_2 lie continually on two fixed lines u_1 and u_2, while the vertex S is a fixed point and the angle A_1SA_2 is of constant magnitude. Show that the base A_1A_2 generates a pencil of rays of the second order to which u_1 and u_2 belong.

6. If two pairs of rays a_1, b_1, and a_2, b_2, lying in the same plane, rotate about fixed points, the first pair about S_1 and the second pair about S_2, while the angles (a_1b_1) and (a_2b_2) are of fixed magnitude and one point of intersection (a_1a_2) traverses a straight line, show that the remaining points of intersection (a_1b_2), (a_2b_1), (b_1b_2), describe curves of the second

order passing through S_1 and S_2. (Newton, "The Organic Development of a Conic," *Principia*, Bk. I, Lemma XXI.)

7. The base A_1A_2 of a variable triangle A_1PA_2 is of fixed length and slides along a fixed straight line u while the sides A_1P and A_2P rotate about fixed points S_1 and S_2, respectively. Show that the vertex P describes a hyperbola passing through S_1 and S_2 of which the line u is an asymptote.

8. If S is any point from which the vertices of a simple plane quadrangle are projected by harmonic rays, the locus of S is a curve of the second order passing through those vertices.

9. If $ABCD$ is a complete quadrangle whose sides AB, AC, AD, BC, BD, CD, are cut by an arbitrary straight line in the points P, Q, R, S, T, V, respectively, and if E, F, H, K, L, M, are the harmonic conjugates of these points relative to the two vertices of the quadrangle on the same side, show that a curve of the second order will pass through the six points E, F, H, K, L, M on which will lie the points of intersection X, Y, Z of pairs of opposite sides of the quadrangle. (*Annals of Mathematics*, VII, p. 73.)

Suggestion. The sets of points E, F, S; E, H, T; and similarly situated sets, are collinear. (Exercise 8, p. 39.) Hence in the hexagon $EFHMLK$, the pairs of opposite sides intersect in the points S, V, Q, which are collinear. Consequently, the hexagon is inscriptible in a curve of the second order (§ 68). Also, if X is the point of intersection of the pair of opposite sides AB and CD, the hexagon $EHFKMX$ is inscriptible in a curve of the second order having five points in common with this same curve.

10. State and prove the plane reciprocal of Exercise 9.

11. If two concentric pencils of rays lying in different planes which intersect obliquely are so correlated that each ray of one pencil is perpendicular to its homologous ray in the other, the planes determined by pairs of homologous rays form a pencil of the second order.

12. If a plane cuts the six edges of a tetrahedron $ABCD$ in the points P, Q, R, S, T, V, respectively, and the harmonic conjugates of these points relative to the two vertices on the same edge are E, F, H, K, L, M, these six points are projected from any point O of the plane by rays of a cone of the second order on which will lie also the three rays through O drawn to meet a pair of opposite edges of the tetrahedron.

The proof of this theorem is analogous to that of Exercise 9.

CHAPTER VII

RULED SURFACES OF THE SECOND ORDER

75. Additional Forms of the Second Order. To complete the enumeration of forms derived from two projectively related primitive forms of the first order, we must still consider those which may be generated by two projectively related ranges of points not lying in the same plane and by two projectively related pencils of planes whose axes do not intersect.

76. Regulus of the Second Order generated by two Projective Ranges of Points. Suppose there are given two projective ranges of points u_1 and u_2 which do not lie in the same plane and which consequently do not intersect. If A_1, B_1, C_1, D_1, $\cdots$ and A_2, B_2, C_2, D_2, $\cdots$ are homologous points in these two ranges, the rays A_1A_2, B_1B_2, C_1C_2, $\cdots$ will form a sequence of rays or a ***regulus*** which lies on a ***ruled surface.*** The surface is said to be ruled because it may be traversed by a moving straight line which lies wholly on it.

Any straight line u which meets three of the rays A_1A_2, B_1B_2, C_1C_2, $\cdots$ $(v_1, v_2, v_3, \cdots)$ will intersect all such rays. For if u is taken as the axis of coincident pencils of planes (Fig. 45) projecting the ranges of points $A_1B_1C_1\cdots$ and $A_2B_2C_2\cdots$ respectively, these two pencils of planes will be projectively related and will have three planes self-corresponding; namely, the planes uv_1, uv_2, and uv_3. Hence all planes of these superposed pencils will be self-corresponding (§ 54) and every plane through u will cut u_1 and u_2 in homologous points. The line joining any two homologous points of u_1 and u_2 will lie, therefore, in a plane of the pencil u and will

intersect the axis u. That is to say, any line u which intersects three of the v-lines intersects all such lines.

Since a line u may be drawn through any point of v_1 to meet v_2 and v_3, and consequently to meet all v-lines, there is not only an unlimited series of lines $v_1, v_2, v_3, \cdots$ lying on the ruled surface determined by the projective ranges u_1 and u_2, but also an unlimited series of lines $u_1, u_2, u_3, \cdots$ lying on that surface.

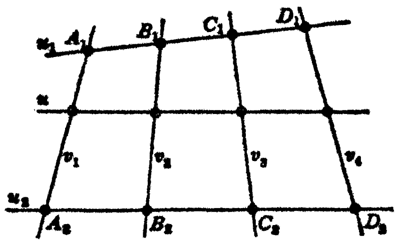


FIG. 45

Of the lines $v_1, v_2, v_3, \cdots$ no two can intersect, for in that case they would lie in the same plane, as would also u_1 and u_2, contrary to assumption. Neither can any two of the lines $u_1, u_2, u_3, \cdots$ intersect, but every u-line intersects every v-line and at every point of the surface on which the two systems lie there is a ray of each system. Since a v-line intersects a u-line at every point, and in particular, at its infinitely distant point, to every u-line there is a parallel v-line; and, likewise, to every v-line there is a parallel u-line.

An arbitrary straight line may be drawn to meet at most two rays of either regulus without meeting all rays of that regulus, that is, without lying wholly on the surface, and for that reason the regulus is said to be of the second order and the surface on which the regulus lies is called a ruled surface of the second order, or a ruled quadric surface. Hence we may state the following theorem.

THEOREM. *Two projectively related ranges of points which do not lie in the same plane generate a regulus of the second order which lies on a ruled surface of the second order. On this surface there lies also another regulus of the second order, and these two are so related that each ray of the one intersects every ray of the other while no two rays of the same regulus intersect.*

77. Regulus of the Second Order generated by two Projective Pencils of Planes. Suppose, on the other hand, there are given two projectively related pencils of planes whose axes u_1 and u_2 do not intersect, and that of these projective pencils $\alpha_1, \beta_1, \gamma_1, \cdots$ and $\alpha_2, \beta_2, \gamma_2, \cdots$ are homologous planes. The lines of intersection of pairs of homologous planes, $\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2, \cdots$ $(v_1, v_2, v_3, \cdots)$, will meet both axes u_1 and u_2, and will join pairs of homologous points in projective ranges on these lines, since the axis u_1 meets the planes of the pencil u_2, and the axis u_2 meets the planes of the pencil u_1, in ranges of points projectively related.

Hence, the system of rays generated by the projective pencils of planes u_1 and u_2 is identical with the system of rays generated by the projective ranges of points u_1 and u_2, in § 76, and the properties of the two reguli so generated are the same.

78. Any two Rays of Either Regulus on a Ruled Quadric may be taken as the Bases of the Generating Forms. The range of points u_2 and the range of corresponding points on any other u-line, u_k say, are both perspective to the pencil of planes u_1 projecting the v-lines, those points of u_2 and u_k being homologous which lie in the same plane of u_1, that is, which lie on the same v-line. From this it follows that the regulus of v-lines lying on a ruled surface of the second order is cut by any two of the u-lines in projective ranges of points. Similarly, since the pencil of planes u_2

and the pencil of corresponding planes through any other u-line are both perspective to the range of points u_1 determined by the v-lines, it is easy to see that the v-lines are projected from any two of the u-lines by projective pencils of planes.

Any two rays of the one system may be called *directors* of the other system and each regulus is the director regulus of the other. A single ray may be spoken of as a *generator* of the regulus to which it belongs, or of the surface, since the surface is traversed by the motion of any ray of either regulus lying on it.

79. The Lines meeting three Arbitrary Rays lie on a Quadric Surface. The system of lines meeting three arbitrary rays in space, no two of which lie in the same plane, is a regulus of the second order. For if two of these lines are taken as axes of pencils of planes projecting the points of the third line, those being homologous planes which project the same point, these two pencils of planes are projectively related and the lines of intersection of pairs of homologous planes will form the system of rays meeting the three given lines. The system is therefore a regulus of the second order and lies on a quadric surface.

80. Tangent Planes of a Ruled Quadric. At any point of a ruled surface of the second order, a u-line and a v-line intersect (§ 76) and the plane of these two lines intersects all other rays of the two reguli. At the point in which the u-line and the v-line intersect, the plane is said to be *tangent to* the surface, and the point is the *point of contact* of the plane. At every point of the surface there is, therefore, a tangent plane cutting the surface along two straight lines. Every line of the tangent plane meets the surface at two points; namely, the points in which it intersects the u-line and the v-line lying in the plane, except that lines of the

plane through the point of contact, other than the u-line and the v-line, meet the surface only at that point.

Every plane through a u-line (or a v-line) on the surface contains also a v-line (or a u-line) and is therefore tangent to the surface at some point along the line. As the plane rotates about u, the point of contact moves along u marking out a range of points projective to the pencil of planes formed by the rotating plane. For the pencil of planes so generated cuts any other u-line on the surface in a range of points projective to the range on u in which the v-lines intersect it.

81. Plane Sections of a Ruled Quadric. A plane section of a ruled surface of the second order consists either of two straight lines or else it is a curve of the second order.

Any plane which contains a ray of one regulus on the ruled surface (a u-line, say) contains also a ray of the other regulus (a v-line) and is tangent to the surface (§ 80). Along the u-line the tangent plane cuts the rays of the v-regulus and along the v-line it cuts the rays of the u-regulus. The section of the surface therefore by a tangent plane consists of two straight lines on which lie two ranges of points of the first order.

If, however, an arbitrary plane contains no ray of either regulus, it will intersect the two projective pencils of planes generating the surface in two projective pencils of rays. These pencils determine a curve of the second order in the points of which the plane intersects a ray of each regulus lying on the surface. The section of the surface therefore by an arbitrary plane is a curve of the second order.

82. Projections of a Regulus of the Second Order from an Arbitrary Point. The systems of rays lying on a ruled surface of the second order are projected from a given point either by two pencils of planes of the first order or by a pencil of planes of the second order.

If the given point lies on the surface, both a u-ray and a v-ray of the surface pass through it. The rays of the u-regulus lying on the surface, projected from the point, determine a pencil of planes of the first order having the v-ray through the point as axis, while the rays of the v-regulus determine a pencil of planes having the u-ray through the point as axis. Thus, the rays of the surface projected from a given point on the surface determine two pencils of planes of the first order whose axes intersect at the point.

If the given point does not lie on the surface, the two ranges of points u_1 and u_2 which generate the surface, projected from the given point, determine two concentric pencils of rays, projectively related, lying in different planes. These pencils of rays generate a pencil of planes of the second order (§ 62), each plane of which contains a v-ray, and likewise a u-ray, of the surface.

It should be noted that in the first of these two cases, the projecting planes are each tangent to the surface at some point along the axes of the two pencils of planes; and in the second case, each projecting plane is tangent to the surface at the point of intersection of the two rays of the surface which lie in it.

83. The Tangent Planes at points of a Plane Section form a Pencil of the Second Order. If a section of a ruled surface of the second order is made by an arbitrary plane containing no ray of the surface, the tangent planes at the points of section form a pencil of planes of the second order; and, reciprocally, the points of contact of the planes projecting a ruled surface of the second order from any point not on the surface, lie on a curve of the second order.

First, take any three points A, B, C, on the given section of the surface made by a plane σ. The tangent planes at these points meet in a point P from which the rays of the surface are projected by a pencil of planes of the second

order (§ 82). The section of this pencil of planes made by the plane σ is a pencil of rays of the second order (§ 62) whose points of contact form a curve of the second order [1] having in common with the curve of section of the surface the points A, B, C, and the tangents at those points. The two curves and their tangents, therefore, are identical and the tangent planes at the points of the curve of section coincide with the planes of the pencil of the second order.

Reciprocally, if the pencil of planes projecting the rays on a ruled surface of the second order is cut by the plane determined by the points of contact of three of these planes, the section is a pencil of rays of the second order of which the points of contact form a curve of the second order (§ 92) identical with the section of the surface made by this plane.

84. A Ruled Surface of the Second Order is also of the Second Class. A given line which has more than two points in common with a ruled surface of the second order lies wholly on the surface and every plane through it is tangent to the surface at some point (§ 80). If, however, a given line k has two distinct points A and B in common with the surface, and these two points only, at both A and B the line meets a generator of each regulus lying on the surface, u_a and v_a at A, u_b and v_b at B. The plane of k and u_a will contain the generator v_b since u_a and v_b must intersect, and it will be tangent to the surface at the intersection of these two generators. Similarly, the plane of k and v_a will contain the generator u_b and will be tangent to the surface at the intersection of v_a and u_b. No other plane through k can be tangent to the surface since no other plane can contain a generator.

If the line k should meet the surface but once, that is, should be tangent to the surface at a point K, it would lie in

[1] This statement is made at this point for completeness in the treatment of these ruled surfaces, though its proof is deferred to § 92.

the tangent plane at K and no other plane through it could be tangent to the surface. And further, if the line k does not meet the surface, no plane through it will be tangent to the surface since no plane through the line contains a generator. Since not more than two planes tangent to a surface of the second order can pass through any line not lying wholly on it, it follows that the surface of the second order is also of the second class.

85. Classification of Ruled Surfaces of the Second Order. The system of v-lines lying on a ruled surface of the second

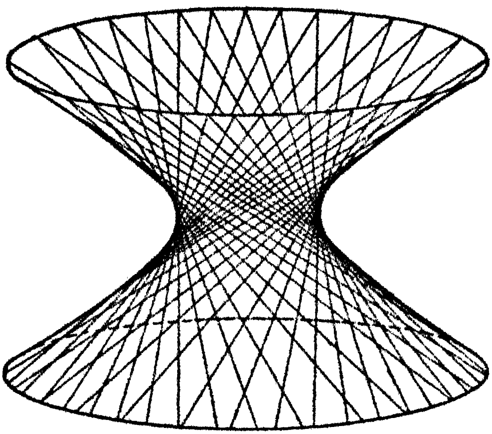


FIG. 46

order, generated by two projectively related ranges of points u_1 and u_2, will take on essentially different properties according as the infinitely distant point of u_1 is or is not homologous to the infinitely distant point of u_2.

If the infinitely distant points of u_1 and u_2 are not homologous, all the rays of the v-system, and likewise all the rays of the u-system, come into the finite region; in other words, there is no infinitely distant ray belonging to either regulus.

The surface in this case is called a ***hyperboloid of one sheet***, or a ***simple hyperboloid*** (Fig. 46).

If, on the other hand, the infinitely distant point of u_1 is homologous to the infinitely distant point of u_2, there is a v-line which lies wholly at infinity. Every plane through this v-line contains also a u-line, and in particular the infinitely distant plane contains both a u-line and a v-line and is therefore tangent to the surface at their intersection. The surface in this case is called a ***hyperbolic paraboloid*** (Fig. 47).

Any plane tangent to a hyperboloid of one sheet cuts the surface along two actual lines, and a finite plane parallel to a

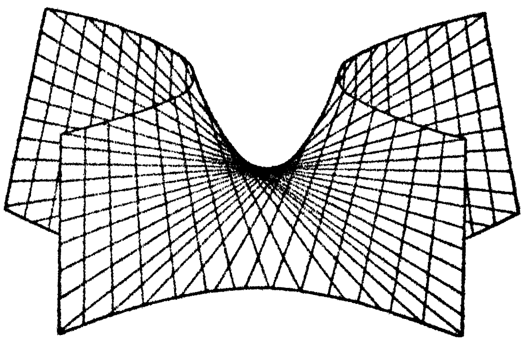


FIG. 47

tangent plane cuts the surface in a curve of the second order having two distinct infinitely distant points, that is, it cuts it in a hyperbola. If, however, the point of contact of the tangent plane is infinitely distant, that is, if the u-line and the v-line in the tangent plane are parallel, the curve of section of a parallel plane is a parabola.

Planes not parallel to tangent planes of a hyperboloid of one sheet cut all rays of either regulus on the surface in finite points, hence in ellipses, while the infinitely distant plane of space cuts the surface in an infinitely distant curve of the

second order. The tangent planes at the infinitely distant points of a hyperboloid of one sheet form a pencil of planes of the second order (§ 83) enveloping a cone of the second order, the so-called *asymptotic cone*. Any plane parallel to one ray of the asymptotic cone cuts the surface in a parabola, a plane parallel to two rays of this cone cuts the surface in a hyperbola, while a plane parallel to no ray of the cone cuts the surface in an ellipse.

A section of a hyperbolic paraboloid is a hyperbola or a parabola according as the plane of section cuts the two rays of the surface lying in the infinitely distant plane, in two distinct points, or passes through their intersection. That is to say, planes parallel to a straight line having a particular direction intersect a hyperbolic paraboloid in parabolas, while planes not parallel to this line intersect the surface in hyperbolas.

86. Special Properties. If straight lines are drawn through a given point parallel to the rays of one regulus of a ruled surface of the second order, these all lie in one plane or on a cone of the second order according as the surface is a hyperbolic paraboloid or a hyperboloid of one sheet.

For, the infinitely distant points of one regulus of a hyperbolic paraboloid lie on a straight line, while in a hyperboloid of one sheet, the infinitely distant points of one regulus lie on a curve of the second order.

Of a hyperbolic paraboloid, the rays of the u-system all meet an infinitely distant v-line, that is, they are parallel to a given plane; and, likewise, the rays of the v-system are parallel to a second plane. If these two planes are perpendicular to each other, the paraboloid is said to be *rectangular*. In a rectangular paraboloid, there is one ray of either regulus which is perpendicular to the directing plane of the other regulus, and consequently there is one ray which is perpendicular to all the rays of the other regulus.

EXERCISES

1. Given two points P_1 and P_2 on a ruled surface of the second order, not on the same ray of either regulus. The rays on the surface through these points intersect in two other points of the surface, Q_1 and Q_2, and the faces of the tetrahedron $P_1Q_1P_2Q_2$ are each tangent to the surface. The tetrahedron is thus both inscribed and circumscribed to the surface.

2. The three principal diagonals of a skew hexagon lying on a ruled surface of the second order intersect in one point.

3. If a range of points u and a pencil of rays of the first order S, not lying in the same or in parallel planes, are related projectively, and rays are drawn through the points of u parallel to the corresponding rays of S, they will form one regulus of a hyperbolic paraboloid.

4. If a range of points u and a pencil of planes v whose bases are not at right angles, are related projectively, and perpendiculars are drawn from the points of u to the corresponding planes of v, they will form one regulus of a hyperbolic paraboloid.

5. If at the points of a straight line lying on a ruled surface of the second order, perpendiculars are drawn to the surface, they will form one regulus of a hyperbolic paraboloid.

6. The planes through a given point normal to the rays of one regulus of the second order form a pencil of planes of the first or the second order according as the surface on which the regulus lies is a hyperbolic paraboloid or a hyperboloid of one sheet.

7. Construct a ruled surface of the second order of which there are given two rays a and b, not lying in the same plane, and either three points outside a and b, or three tangent planes not passing through a or b.

8. The sides of a fixed skew hexagon are tangent to a cone of the second order; show that the locus of the vertex of this cone is a ruled surface of the second order on which the three diagonals of the hexagon lie.

CHAPTER VIII

DEDUCTIONS FROM PASCAL'S AND BRIANCHON'S THEOREMS

87. Pascal's Theorem applied to an Inscribed Pentagon. Interesting results are derived from Pascal's and Brianchon's theorems when certain modifications are made in the diagrams illustrating them.

For example, if $ABCDEF$ is any simple hexagon inscribed in a curve of the second order (Fig. 48), the pairs of opposite sides, AB and DE, BC and EF, CD and FA, intersect at points P, Q, R, respectively, of one straight line (Pascal's theorem). Moreover, the pencils of rays projecting the four remaining vertices of the hexagon from the centers A and C are projectively related (§ 66). That is,

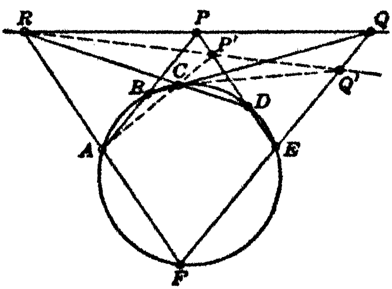


FIG. 48

$$A(BDEF) \barwedge C(BDEF).[1]$$

Suppose now that the vertex B of the hexagon moves along the curve towards C, while the other vertices remain fixed and consequently the point R of the Pascal line remains fixed. The side AB will rotate about the vertex A and will mark out a range of points on the side DE, and the

[1] The symbol $\barwedge$ is an abbreviation for "is projective to," so that $A(BDEF) \barwedge C(BDEF)$ is read, "the pencil $A(BDEF)$ is projective to the pencil $C(BDEF)$."

The symbol $\doublebarwedge$ is used as an abbreviation for "is perspective to."

side CB will rotate about the vertex C and will mark out a range of points on the side EF. Since the pencil of rays A is projective to the pencil of rays C, the range of points on DE is projective to the range of points on EF. The Pascal line PQR will rotate about the fixed point R, and will join pairs of homologous points in the ranges DE and EF, these ranges being perspective since their common point E is self-corresponding (§ 55).

When the point B comes to coincide with the point C, the ray AB coincides with AC, and the ray CB takes the position of the tangent at C (§ 64). The six sides of the inscribed hexagon are then the sides of an inscribed pentagon and a tangent at one vertex. In this position the Pascal line passes through the fixed point R, through P', the intersection of AC and DE, and through Q', the intersection of EF and the tangent at C. For an inscribed pentagon, therefore, Pascal's theorem may be stated in the following form.

THEOREM. *If a simple pentagon is inscribed in a curve of the second order, two pairs of non-adjacent sides intersect in points of a straight line on which also intersect the fifth side and the tangent at the opposite vertex.*

In the process by which the side AB of the inscribed hexagon became the side AC of the pentagon and the side BC of the hexagon took the position of the tangent at C in the pentagon, the tangent may be looked upon as the limiting position of the side BC joining two points of the curve which have come to coincide. In the same way, the point of contact in a ray of a pencil of the second order may be looked upon as the limiting position of the point of intersection of two rays of the pencil which have come to coincide.

88. Brianchon's Theorem applied to a Pentagon in a Pencil of Rays. Suppose $PQRSTV$ is any simple hexagon whose sides a, b, c, d, e, f, are rays of a pencil of the second

order (Fig. 49) so that the lines joining pairs of opposite vertices, PS, QT, RV, pass through one point (Brianchon's theorem). Let the side RQ, always remaining a ray of the pencil, come to coincide with the adjacent side QP, while other rays of the hexagon remain fixed. The vertex Q in its limiting position becomes the point of contact Q' in the side QP and the vertex R takes the position R' on the side SR. The hexagon thus reduces to a pentagon whose sides are rays of a pencil of the second order, the sixth vertex of the hexagon becoming the point of contact in one side. For this figure, Brianchon's theorem may be stated in the following form.

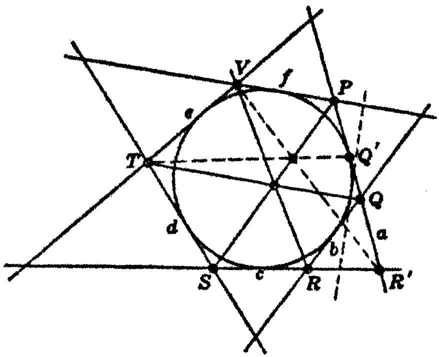


FIG. 49

THEOREM. *If the sides of a simple pentagon are rays of a pencil of the second order, the lines joining two pairs of non-adjacent vertices intersect in a point through which also passes the line joining the fifth vertex and the point of contact in the opposite side.*

89. Pascal's and Brianchon's Theorems applied to Quadrangles, Quadrilaterals, and Triangles. The process of § 87 may be carried further. Not only may the vertices B and C be brought to coincide at C and the side BC take the position of the tangent at C, but two other vertices, E and F, may also come to coincide at the opposite vertex F (Fig. 50), the vanishing side becoming the tangent at F. The hexagon then becomes an inscribed simple quadrangle $ACDF$ with tangents at two opposite vertices, C and F, and

the Pascal line on which these tangents intersect is determined by the points of intersection of the two pairs of opposite sides of the quadrangle AC and DF, CD and AF.

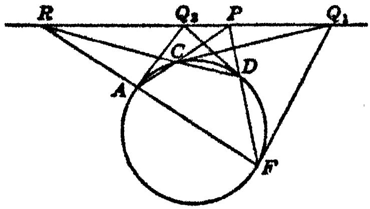


FIG. 50

Or it may be that the two evanescent sides have become tangents at the opposite vertices A and D, in which case the inscribed quadrangle is the same as before and these tangents intersect on the same line as do the tangents at the vertices C and F. Pascal's theorem applied to an inscribed quadrangle, then, takes the following form.

THEOREM. *If a simple quadrangle is inscribed in a curve of the second order, the pairs of opposite sides intersect in points of a straight line on which also intersect the pairs of tangents at opposite vertices.*

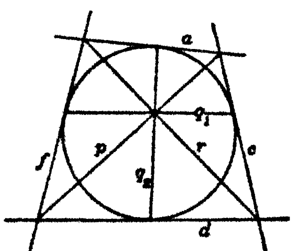


FIG. 51

Brianchon's theorem modified in a similar way so as to be applicable to a simple quadrilateral (Fig. 51) takes the following form.

THEOREM. *If the sides of a simple quadrilateral are rays of a pencil of the second order, the lines joining the pairs of opposite vertices intersect in a point through which also pass the lines joining the points of contact in pairs of opposite sides.*

Similar considerations result in the following forms of Pascal's and Brianchon's theorems as applied to triangles.

THEOREM. *If a triangle is inscribed in a curve of the second order, the three points in which the sides of the triangle intersect the tangents at the opposite vertices lie on one straight line.*

THEOREM. *If the sides of a triangle are rays of a pencil of the second order, the three lines joining the vertices of the triangle to the points of contact in the opposite sides pass through one point.*

90. Direct Proofs of the Quadrangle and Quadrilateral Theorems. The theorems stated above for the quadrangle inscribed in a curve of the second order and the quadrilateral whose sides are rays of a pencil of the second order are of such importance that they merit an independent and more direct proof.

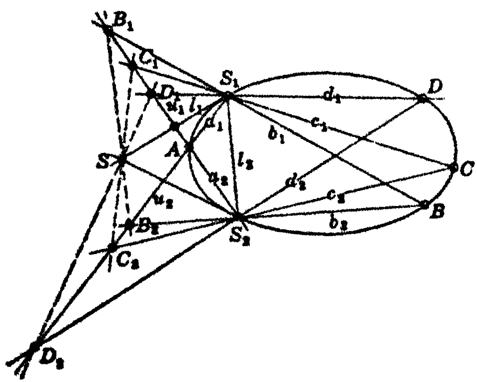


FIG. 52

In the process of finding additional points of a curve of the second order from five given elements (§ 57), the arbitrary lines u_1 and u_2 through the point A may be drawn so as to coincide with a_2 and a_1, respectively, without loss of generality. In that case (Fig. 52), the center S, determined as the point of intersection of B_1B_2 and C_1C_2, is also the point of intersection of the tangents at S_1 and S_2. For the ray S_2S_1, or l_2, intersects u_2 at S_1 and the corresponding ray l_1 of

S_1 is the line S_1S. But this ray is the tangent at S_1 (§ 64). Similarly S_2S is the tangent at S_2.

Through the point S will pass not only the line B_1B_2, that is, the line joining the intersections (b_1a_2) and (b_2a_1), and the line C_1C_2, that is, the line joining the intersections (c_1a_2) and (c_2a_1), but also D_1D_2, or the line joining (d_1a_2) and (d_2a_1), and the lines joining all other such intersections. Also, since any point of the curve other than A might have been chosen to play the same part in the construction for the tangents at S_1 and S_2 (§ 57), it follows that the lines joining the intersections of (b_1c_2) and (b_2c_1), (b_1d_2) and (b_2d_1), (c_1d_2) and (c_2d_1), and all lines similarly drawn, will pass through the point S, the intersection of the tangents at S_1 and S_2.

Moreover, since S_1 and S_2 are not special points of the curve (§ 66), if any simple quadrangle S_1BS_2C is inscribed in a curve of the second order, the intersections of pairs of opposite sides, S_1B and S_2C, or (b_1c_2), S_1C and S_2B, or (c_1b_2), are collinear with the intersection of tangents at the opposite vertices S_1 and S_2. But in the construction the vertices B and C might as well have been chosen as centers of the pencils of rays, and consequently the intersection of tangents at these points is likewise collinear with the points of intersection of pairs of opposite sides.

On the other hand, if in the construction for determining additional rays of a pencil of the second order from five given rays (§ 56), the points S_1 and S_2 are chosen to coincide with A_2 and A_1, respectively, the line of perspectivity, u (Fig. 53), will join the points of contact in the rays u_1 and u_2, and consequently, it is the same whether S_1 and S_2 are chosen to coincide with A_2 and A_1, respectively, or with B_2 and B_1, C_2 and C_1, or with any other pair of homologous points (§ 56). Therefore not only do A_1B_2 and A_2B_1, A_1C_2 and A_2C_1, and other such pairs of rays intersect on the line u, but so also do B_1C_2 and B_2C_1, B_1D_2 and B_2D_1, K_1L_2 and K_2L_1, and all pairs of lines similarly drawn.

If, now, we choose a simple quadrilateral $A_1A_2B_2B_1$ whose sides are rays of the pencil of the second order, the lines joining pairs of opposite vertices A_1B_2 and A_2B_1 intersect in a point of the line u which joins the points of contact in a pair of opposite sides. In the construction, either pair

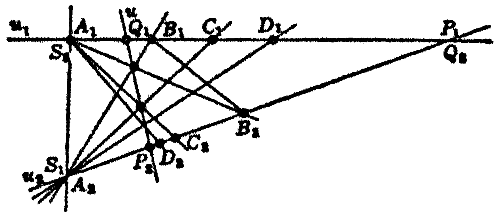


FIG. 53

of sides may be chosen as base lines u_1 and u_2 (§ 66). Consequently, through this same point will pass the line joining the points of contact in the other pair of opposite sides.

91. Relation between a Quadrangle in a Curve of the Second Order and a Quadrilateral in a Pencil of Rays of the Second Order. The theorems on quadrangles and quadrilaterals stated in § 89 and proved independently in § 90, when placed side by side, present an interesting relation.

THEOREM ON QUADRANGLES. *If four points of a curve of the the second order are the vertices of a complete quadrangle and the tangents at those points are the sides of a complete quadrilateral, these two forms are so related that the three pairs of opposite vertices of the quadrilateral lie on the sides of a triangle in whose vertices the three pairs of opposite sides of the quadrangle intersect.*

THEOREM ON QUADRILATERALS. *If four rays of a pencil of the second order are the sides of a complete quadrilateral and their points of contact are the vertices of a complete quadrangle, these two forms are so related that the three pairs of opposite sides of the quadrangle intersect in the vertices of a triangle on whose sides the three pairs of opposite vertices of the quadrilateral lie.*

The complete quadrangle comprises three simple quadrangles for each of which the quadrangle theorem of § 89 is true. The three lines (Pascal lines) arising in the three simple quadrangles are the sides of the triangle in question.

The complete quadrilateral comprises three simple quadrilaterals for each of which the quadrilateral theorem of § 89 is true. The three points (Brianchon points) arising in the three simple quadrilaterals are the vertices of the triangle in question.

In this form, the theorem on the left and the theorem on the right state exactly the same thing; namely, that the given triangle in either case is such that in its vertices the pairs of opposite sides of the complete quadrangle intersect and on its sides the pairs of opposite vertices of the complete quadrilateral lie. It is immaterial, therefore, whether the theorems refer to four points of a curve of the second order and the tangents at those points or to four rays of a pencil of the second order and the points of contact in those rays.

Suppose, then, there are given a complete quadrangle $KLMN$ and a complete quadrilateral $klmn$ whose sides pass through the vertices of the quadrangle (Fig. 54), and that these two forms are so related that the pairs of opposite sides of the quadrangle intersect in the vertices of a triangle XYZ on whose sides the pairs of opposite vertices of the quadrilateral lie. If a curve of the second order is constructed passing through the points $KLMN$ and having the line k for tangent at K, there being only one such curve (§ 67), the lines l, m, n, will be tangents to this curve at the points L, M, N, respectively, by the theorems just stated.

For if l, for instance, is not the tangent at L, but some other line through L is tangent to the curve at that point, the tangents at K and L could not intersect on a side of the triangle XYZ, as must be the case by the theorem on the left.

If, on the other hand, the pencil of rays of the second order is constructed from the elements k, l, m, n, with K as the point of contact in k, then the points L, M, N, will be points of contact in the rays l, m, n, respectively. Consequently, four points of a curve of the second order and

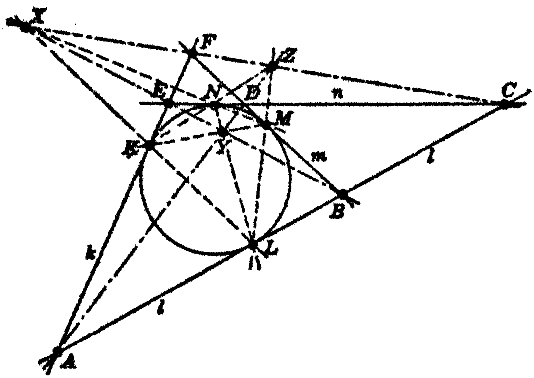


FIG. 54

the tangents at those points are, at the same time, four rays of a pencil of the second order and the points of contact in those rays.

92. The Tangents to a Curve of the Second Order form a Pencil of Rays of the Second Order, and Reciprocally. That the tangents at the points of a curve of the second order form a pencil of rays of the second order, and, reciprocally, that the points of contact in the rays of a pencil of the second order form a curve of the second order, will appear with greater generality from the following direct demonstration.

In the diagram of § 91 (Fig. 54), suppose the points L, M, N, of a given curve of the second order and the tangents at those points remain fixed, while K and its tangent k move along the curve. In every position of K, the figure

$KLMN$ is an inscribed quadrangle with fixed tangents at L, M, and N; consequently, the points B, C, D, the intersections of l and m, l and n, m and n, respectively, are fixed, as are also the lines LM, LN, and MN.

In the simple quadrangle $KMLN$, the pairs of opposite sides intersect in Y and Z, and the pairs of tangents at opposite vertices intersect at A and D, respectively. Also, in the simple quadrangle $KMNL$, the pairs of opposite sides intersect in Y and X, respectively, and the pairs of tangents at opposite vertices intersect at E and B.

Hence, as K moves along the curve, the rays DA and BE will rotate about the fixed points D and B, respectively, generating perspective pencils of rays of which homologous rays intersect in the variable point Y of the fixed line LN. The points A and E, therefore, describe projective ranges of points on the fixed lines l and n, and the tangent k joins pairs of homologous points in these ranges. The tangent therefore generates a pencil of rays of the second order.

On the other hand, suppose in the same diagram (Fig. 54) $klmn$ is a quadrilateral in a pencil of rays of the second order, of which the rays l, m, n, and the points of contact, L, M, N, in these rays remain fixed while k moves, always remaining a ray of the pencil.

The ranges of points described by k on the fixed rays l and n will therefore be projective, and the pencils of rays projecting these ranges from the fixed points D and B will be perspective, since homologous rays intersect in the point Y of the fixed line LN. The ranges of points described by Z and X, respectively, are sections of these pencils and are consequently projective. Hence the pencils of rays generated by NK and LK, projecting these ranges of points, are projectively related. That is, the point of contact K in the ray k traverses a curve of the second order. From this demonstration we have the following theorem.

THEOREM. *If a point moves on a curve of the second order, the tangent to the curve at that point will generate a pencil of rays of the second order; and reciprocally, if a line moves in a pencil of rays of the second order, the point of contact in that line will describe a curve of the second order.*

In other words, the tangents to a curve of the second order form a pencil of rays of the second order and the points of contact in the rays of a pencil of the second order form a curve of the second order. The rays of the pencil may be said to *envelop* the curve.

93. Brianchon's Theorem re-stated for a Curve of the Second Order. Brianchon's theorem may now be stated as follows.

THEOREM. *If the sides of a simple hexagon are tangents to a curve of the second order, the three principal diagonals, that is, the lines joining pairs of opposite vertices, pass through one point.*

The theorems on pentagons, quadrilaterals, and triangles whose sides are rays of a pencil of the second order, may likewise be re-stated in terms of similar figures circumscribing a curve of the second order.

Moreover, since a pencil of rays of the second order is cut by any two of its rays in projective ranges of points, it follows that the tangents to a curve of the second order are cut by any two of these tangents in projective ranges of points.

94. The Points of a Curve of the Second Order are Projective to the Tangents at those Points. In the discussion of § 92, it developed that the two pencils of rays described by BE and DA (Fig. 54) are persepctive, since homologous rays in the two pencils intersect in Y, a point of the fixed line LN. The pencil of rays MK projecting the moving point K of the curve from the fixed point M is per-

spective to both of these pencils and the range of points described by E on the fixed tangent at N is projective to the pencil of rays M since it is a section of the pencil of rays B. The tangent at K passes through E and we have the following interesting property.

THEOREM. *Of a curve of the second order if there are given a fixed point M and a fixed tangent n and to each ray of M projecting a point K of the curve there is correlated that point of n through which the tangent at K passes, then the pencil of rays M is projective to the range of points n.*

95. The Cross-Ratios of Points of a Curve of the Second Order and of the Tangents at those Points are Equal.[1] Four points of a curve of the second order are said to be harmonic when they are projected from any fifth point of the curve by harmonic rays, and four rays of a pencil of the second order are harmonic when they are cut by any fifth ray of the pencil in harmonic points.

Moreover, the cross-ratio of four points of a curve of the second order is defined to be the cross-ratio of the four rays projecting them from any fifth point of the curve; and likewise, the cross-ratio of four rays of a pencil of the second order is the cross-ratio of the four points in which they are cut by any fifth ray of the pencil.

From the theorem of § 94, it follows at once that if four points of a curve of the second order are harmonic the tangents at those points are harmonic. More generally, the cross-ratio of four points of a curve of the second order is equal to the similar cross-ratio of the tangents at those points.

96. Properties of Cones and Pencils of Planes of the Second Order. By projecting the plane figures of this chapter from a point not in the plane, analogous properties

[1] This property was stated by Chasles in his *Traité des Sections Coniques*, 1865.

of cones of the second order and pencils of planes of the second order will be obtained. For example, the tangent planes of a cone of the second order form a pencil of planes of the second order and the rays of contact in a pencil of planes of the second order form a cone of the second order.

The properties in a cone of the second order analogous to Pascal's and Brianchon's theorems may be stated as follows.

THEOREM. *If the vertices of any simple hexagon lie on a cone of the second order, the planes determined by the vertex of the cone and the three pairs of opposite sides of the hexagon intersect in three rays through the vertex which lie in one plane; and if the sides of any simple hexagon are tangent to a cone of the second order, its three principal diagonals are intersected by one straight line through the vertex of the cone.*

Four rays of a cone of the second order determine three pairs of planes which intersect in three rays through the vertex. These rays lie by twos in three planes, in lines of which the planes tangent to the cone along the four given rays intersect, two and two.

97. A Curve of the Second Order is a Conic Section. We are now ready to show that a curve of the second order is the same as the conic section of ancient geometry.

It has already been shown (§ 73) that a circle is a curve of the second order and hence the projection of a circle from any point not in its plane is a cone of the second order (§ 62). Also any section of a circular cone; that is, a section of a cone which is the projection of a circle, is a curve of the second order. It remains to show that an arbitrary curve of the second order can be made to appear as the section of a circular cone.

Suppose there are given an arbitrary curve of the second order and a circle lying in different planes (Fig. 55) but so situated that the line of intersection k of their planes is

tangent to both curves at the same point K. In the line k choose any three points P, Q, R, different from K, and from each of these draw the second tangents to the two given curves, meeting the circle in A, B, C, and the arbitrary curve of the second order in A', B', C', respectively.

The planes PAA', QBB', RCC', cannot intersect in one line for then PA, QB, and RC would intersect that line where

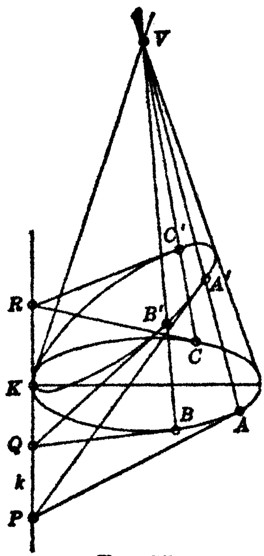


FIG. 55

it meets the plane of the circle and PA', QB', and RC' would intersect the line where it meets the plane of the conic. But these conditions cannot exist since PA, QB, RC, and likewise PA', QB', RC', are tangents to a curve of the second order. They will therefore determine a point V from which the systems of tangents to the two given curves may be projected by two pencils of planes of the second order. These pencils will have four planes in common, namely, the three planes determining V and the plane Vk.

The ray of contact in Vk is also common to the two pencils. The pencils of planes, therefore, are identical and they envelop a cone of the second order on which the two given curves lie (§ 96).

Any curve of the second order may thus be made to lie on a cone which is the projection of a circle; hence it is identical with the section of the cone made by its plane. A curve of the second order may therefore be designated hereafter as a *conic section* or more briefly as a *conic*.[1]

98. Classification of the Sections of a Cone. A plane σ through the vertex of a cone of the second order may intersect all the rays of the cone at the vertex, or it may contain one ray of the cone cutting all the others at the vertex, or it may contain two rays of the cone.

A plane parallel to σ in the first case would cut all rays of the cone on one side of the vertex and the section would have no infinitely distant point. The section would thus be an ellipse (§ 72).

In the second case, a plane parallel to σ would cut one ray of the cone at an infinitely distant point and the other rays on one side of the vertex. The section in this case would be a parabola.

In the third case, a plane parallel to σ would cut two rays of the cone at infinitely distant points and the section would therefore have two points in common with the infinitely distant line of the plane. Of the other rays of the cone, the plane would cut some on one side of the vertex and some on the other. The section, therefore, would be a hyperbola and would consist of two branches connecting through its infinitely distant points.

[1] Conic sections were first studied as plane sections of a right circular cone with an acute, right, or obtuse vertical angle, the ellipse coming from the first of these, the parabola from the second, and the hyperbola from the third. Later, it was shown by Apollonius that any of the three forms might be cut from a single cone with a circular base, whether right or oblique.

EXERCISES

1. Make use of Pascal's and Brianchon's theorems to solve the following problems:

(a) Having given a pentagon inscribed in a conic, draw the tangents at the vertices.

(b) Having given a pentagon circumscribed to a conic, find the points of contact in the sides.

(c) Given four tangents to a parabola, (1) find the point of contact in one of them; (2) draw the second tangent through a given point in one of them.

(d) Given one point and the asymptotes of a hyperbola, find where a given line through the point will intersect the hyperbola a second time.

2. Given three points, A, B, C, of a curve of the second order and the tangents at B and C. If an arbitrary line is drawn through B or through C, make use of Pascal's theorem to find where the line will intersect the curve a second time.

3. Given three rays, a, b, c, of a pencil of the second order and the points of contact in b and c. If an arbitrary point is chosen on b or on c, make use of Brianchon's theorem to draw the second ray of the pencil passing through that point.

4. If a range of points and a pencil of rays lying in the same plane are projectively related and through each point of the range a line is drawn parallel to the homologous ray of the pencil, these will either intersect in one point or they will envelop a parabola.

5. In Exercise 4, if the lines are drawn from the points of the range perpendicular to the homologous rays of the pencil, show that the same result is obtained.

6. An angle of given magnitude so moves in its plane that its vertex describes a straight line u while one side rotates about a fixed point. Show that the other side will envelop a parabola to which u is tangent.

7. If the vertices of a simple hexagon $AC_1BA_1CB_1$ lie alternately on two straight lines in a plane, the intersections of pairs of opposite sides lie on a third straight line. (Pappus, *Mathematicae Collectiones*, VII.)

8. State and prove the plane reciprocal of Exercise 7.

CHAPTER IX

THE THEORY OF POLES AND POLARS

99. Definition of the Pole and Polar Relation. The theorems on inscribed and circumscribed quadrangles (§ 89) lead immediately to other important properties of the curve of the second order.

DEFINITION. Through a point P in the plane of a given conic two secants are drawn at random which cut the conic in the points A, D, and B, C, respectively (Figs. 56, 57). If the lines AB and CD intersect at Q and the lines AC and BD intersect at R, the line QR is called the *polar* of P with respect to the given conic and P is called the *pole* of QR.

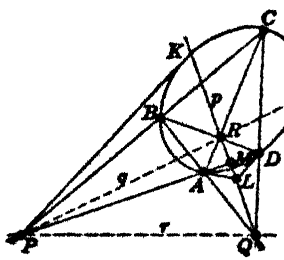


FIG. 56

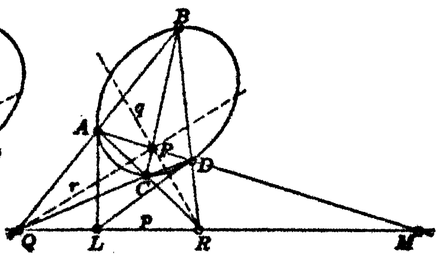


FIG. 57

From this definition it will appear at once that the line PR is the polar of Q, and likewise PQ is the polar of R for the given conic; also that Q is the pole of PR, and R is the pole of PQ. The same relation is expressed when we say that in any complete quadrangle whose vertices lie on a conic each diagonal point is the pole, relative to the conic, of the line determined by the other two.

100. The Pole and Polar Relation is Unique. At first glance it might seem that for any given point there would be many polars since the lines through P cutting the curve are drawn wholly at random, but a study of the construction will show that for any given conic there is but one polar of a given point.

From Pascal's theorem as applied to the inscribed quadrangle $ABCD$ (Figs. 56, 57) the tangents to the curve at A and D intersect on the line QR, say at L, and if M is the point of intersection of QR and AD, then M is the harmonic conjugate of P relative to A and D (§ 33). Consequently, since the polar QR passes through L and M, it may be determined from but one of the secants through P, and no matter how the second secant is drawn we arrive always at the same polar. Moreover, since the polar QR may be determined in the same way from any one of the second secants, the first secant may be drawn wholly at random, and consequently the same polar is determined no matter how the two secants are drawn.

If, then, any number of secants are drawn through P and their points of intersection with the curve are joined, two and two, as in the definition of a polar (Fig. 58), all such pairs of joined lines will intersect on the polar; also the tangents to the curve at points lying on the same secant will intersect on the polar.

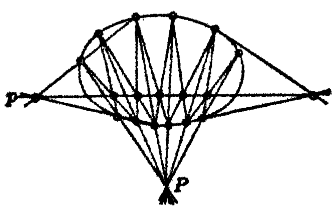


FIG. 58

Since P is harmonically separated from M (Figs. 56, 57) by the points in which the secant AD intersects the curve, and since AD is any secant whatsoever through P, it follows that the polar of P is the locus of points harmonically separated from P by the curve.

101. The Polar is the Chord of Contact of Tangents from the Pole. If the polar p cuts the curve at a point K (Fig. 56), the line PK is a secant through P, and a point K on the polar is the harmonic conjugate of P relative to the two intersections of PK with the curve (§ 100). Since one of these intersections coincides with K, so must also the other (§ 30). Hence PK is tangent to the curve, and the polar of P passes through the points of contact of tangents from P, if such can be drawn.

102. Construction for the Pole of a Given Line. On the other hand, if there is given in the plane of a conic any straight line r (Fig. 59), its pole R relative to the given conic

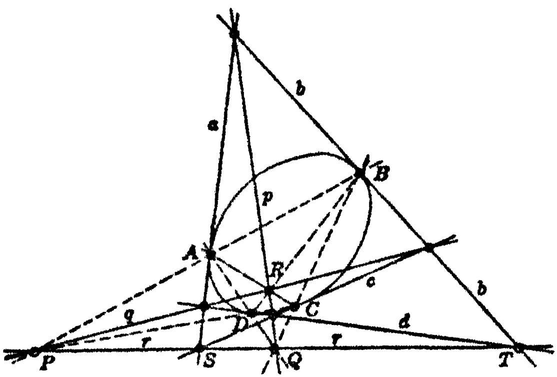


FIG. 59

may be found by choosing at random two points S and T on the given line from which tangents to the conic, a and c from S and b and d from T, can be drawn. The intersections of a and b, c and d; b and c, a and d; determine two lines p and q, which intersect in R, the pole of the given line.

That the point R determined by this construction is the pole of the line r as defined in § 99 may be shown by the following considerations.

if A, B, C, D are the points of contact of the tangents a, b, c, d, respectively, we know from Brianchon's theorem that the lines AC and BD both pass through R, the intersection of the diagonals p and q of the circumscribed quadrilateral $abcd$. Also, the lines AB and CD pass through P, the intersection of q and r; and likewise, AD and BC pass through Q, the intersection of p and r. The polar of R, therefore, by the definition of § 99, is the line PQ, or the given line r. That is, the point R determined in the foregoing construction is the pole of the given line.

It will be observed that the construction here given to determine the pole of a given line relative to a fixed conic is the reciprocal of that given in § 99 to determine the polar of a given point.

103. Properties of Pole and Polar. The relations developed in the preceding paragraphs between a point and its polar relative to a given conic may be summarized in the following reciprocal statements, those statements in the parallel columns being reciprocals which bear like numbers.

On the polar of a point P, for a given conic, there will lie:	Through the pole of a line p, for a given conic, there will pass:
(1) The point on any secant through P which is harmonically separated from P by the curve.	(1) The line drawn from any point of p which is harmonically separated from p by tangents to the conic from that point.
(2) The points of contact of tangents from P, if such tangents can be drawn.	(2) The tangents at the points of intersection of p with the curve, if such there are.
(3) The intersection of tangents to the curve at the points in which any secant through P cuts the curve.	(3) The line joining the points of contact of tangents from any point of p.

104. The Inside and the Outside of a Conic. A curve of the second order may be traversed from any one point of it to any other point of it, or again to the same point, in either sense, without leaving the curve, and it is said therefore to be a *closed curve*. It divides the plane in which it lies into two parts such that you may pass from any point in one part to any other point in the same part, but to no point in the other part, without crossing the curve. The points of one part of the plane are said to lie *inside* the curve, while the points of the other part lie *outside* the curve.

That part of the plane is defined to be the inside of a conic in which, if any point is chosen, all straight lines through it will cut the curve in two points. From a point inside the conic no tangents can be drawn.

Through a point outside a conic some straight lines may be drawn which cut the conic while others do not. Through any point outside the curve two tangents to the conic may be drawn.

If a point P lies inside a conic, the line through it and any point P' cuts the conic in two points, and if P' lies on the polar of P relative to the conic, it is harmonically separated from P by the curve (§ 100). Consequently, all points of the polar of P must lie outside the conic. Moreover, all points harmonically separated from P by the curve lie on the polar.

If a point P lies outside a conic, its polar p with respect to that curve will cut the curve in two points; namely, in the two points of contact of tangents from P (§ 103). The polar will thus lie partly inside the curve and partly outside. The points of the polar which lie inside are harmonically separated from P by the curve; and conversely, all points harmonically separated from P by the curve lie on the polar.

On the other hand, if a line p lies wholly outside a conic, its pole P with respect to that conic lies inside the conic and

is harmonically separated from every point of p by the curve. If a straight line p cuts a conic, its pole P with respect to that conic lies outside and is harmonically separated by the conic from such points of p as lie inside.

If a point P lies on a given conic, the construction defining its polar (§ 99) yields as a limiting figure, the tangent at P as the polar; and, reciprocally, if a given line p is tangent to the curve, the point of contact is its pole. Relative to a given conic, then, every point of the plane has one and only one polar and every line of the plane has one and only one pole. Thus, by the aid of a conic, a one-to-one correspondence is established between the points and the lines of a plane field.

105. Reciprocal Relations between Points and their Polars.

THEOREM. *If a point Q lies on the polar of another point P, relative to a given conic, then P lies also on the polar of Q.*

THEOREM. *If a line q passes through the pole of another line p, relative to a given conic, then p passes also through the pole of q.*

To demonstrate the theorem on the left, three cases must be considered.

First, suppose the point P lies inside the conic. Then its polar lies wholly outside and every point of it is harmonically separated from P by the curve. Since Q is a point on the polar of P, it lies outside the curve and the two points P and Q are harmonically separated by the curve. The polar of Q passes through all points harmonically separated from Q by the curve (§ 104) and hence it passes through P.

Next, suppose P lies outside the conic. Then its polar must cut the conic and some points of it lie inside the curve and some lie outside. If Q lies on the polar of P, whether

inside or outside the curve, the polar of P is a secant through Q and the polar of Q will pass through the common point of the tangents at the curve-points on the secant (§ 103). But these tangents intersect at P. Hence the polar of Q will pass through P.

Finally, suppose P lies on the curve; then its polar is the tangent at P. If Q lies on this tangent, its polar passes through the point of contact, that is, through the given point P.

The theorem on the right follows immediately from that on the left by reciprocation, or it may be proved independently in an entirely similar manner.

106. Polar Reciprocal Figures. The principle of reciprocity or duality which heretofore has rested on certain observations on the properties of points, lines, and planes, and their mutual relations, now takes definite form, at least for the plane, since by means of the polar theory, to every point of a plane there is correlated a definite line; namely, its polar with respect to a fixed conic; and to every line of the plane there is correlated a fixed point; namely, its pole.

To the points of a straight line are correlated the polar rays of those points, all of which pass through the pole of the given line. In other words, the polar reciprocal of a range of points, relative to a given conic in its plane, is a pencil of rays whose center is the pole of the line on which the range lies. The polar reciprocal, relative to a given conic, of a triangle consisting of three points and the lines through them, is a triangle consisting of three lines and their points of intersection. The polar reciprocal of a complete quadrangle is a complete quadrilateral.

It was by means of the polar theory that Brianchon deduced his theorem from that of Pascal. By constructing the tangents at the vertices of Pascal's inscribed hexagon he formed a circumscribed hexagon, every vertex of which was

the pole, relative to the conic, of a side of the inscribed figure, while the intersections of pairs of opposite sides of the latter were the poles of the principal diagonals of the former. Since, by Pascal's theorem, the three intersections of pairs of opposite sides of the inscribed hexagon lie on a straight line, the three diagonals of the circumscribed figure must pass through one point; namely, through the pole of the Pascal line.

This one application of the polar theory will serve to show how, with the help of a conic, for any figure in a plane, another figure can be constructed whose properties may be deduced from known properties of the given figure. Frequent use will hereafter be made of this principle.

107. Conjugate Points and Lines. DEFINITION. If two points P and Q are so situated that each lies on the polar of the other with respect to a given conic, they are said to be *conjugate* to each other relative to that conic; and similarly, two lines are *conjugate* when each passes through the pole of the other.

A point is therefore conjugate to all the points of its polar, and a line is conjugate to all the lines through its pole. In particular, a point on the given conic is conjugate to all the points of the tangent at that point, and a tangent to the given conic is conjugate to all the lines through its point of contact.

A point of the given conic is said to be *self-conjugate* since it lies on its own polar, and a tangent to the given conic is self-conjugate since it passes through its own pole.

Two lines which intersect on a conic are conjugate with respect to that conic only when one of them is tangent to the conic, and two points on a tangent are conjugate only when one of them is the point of contact.

If two points are conjugate with respect to a given conic, their polars are likewise conjugate.

If the line through two points which are conjugate relative to a given conic, cuts that conic, one of the points must lie inside the conic and the other outside, and they are harmonically separated by the points of the conic lying on the line.

If two conjugate lines p and q intersect in a point R outside the conic with respect to which they are conjugate (Fig. 60), one line must cut the conic while the other does not, and they are harmonically separated by the tangents to the conic through their common point (§ 103).

If two conjugate lines intersect in a point R inside the conic (Fig. 61) and cut the conic at A and C, B and D, respectively, these points are projected from any point of

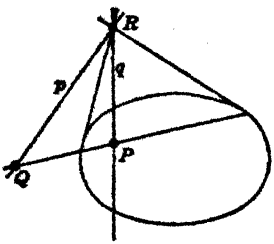


Fig. 60

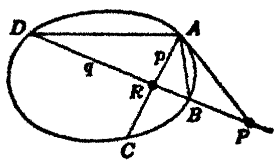


Fig. 61

the conic by harmonic rays. For, the pole of AC lies on BD, say at P, so that the points P, B, R, D, are harmonic (§ 100), and the rays $A(P, B, R, D)$ are likewise harmonic. These rays project the points A, B, C, D of the curve and consequently A, B, C, D are projected not only from A, but from any other point of the curve by harmonic rays (§ 66).

108. Self-Polar Triangles. If P is any point of the plane, Q any point on its polar with respect to a given conic, and R the intersection of the polars of P and Q, the triangle PQR (Fig. 62) is such that each vertex is the pole of the opposite side and each side is the polar of the opposite vertex. Such a

triangle is said to be ***self-polar***, or ***self-conjugate*** relative to the given conic.

Of any self-polar triangle, the two sides through each vertex are conjugate lines relative to the conic and the two vertices on any side are conjugate points. Of such a triangle PQR, not more than one vertex may lie inside the conic, for if a vertex P is inside the conic, its polar p lies wholly outside (§ 104) and consequently the remaining vertices Q and R are outside the conic. Of the two sides of the triangle p and r intersecting at a point Q outside the conic, since they are conjugate, one of them, the side r, say, must cut the conic (§ 107) and the vertex P is harmonically separated from Q by the curve; consequently, it lies inside the curve. That is to say, of the vertices of a self-polar triangle, one vertex lies inside the curve, and the other two, outside. Likewise, two sides of a self-polar triangle relative to a conic intersect the conic, while the third side does not.

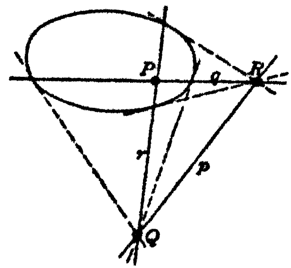


FIG. 62

To construct a self-polar triangle relative to a given conic, one vertex P may be chosen inside the conic and a side q through that vertex may be chosen at random. The polar of P relative to the conic intersects the side q in a point R whose polar r passes through P (§ 105) and intersects p, the polar of P, at a point Q such that the vertices of the triangle PQR are the poles of the opposite sides. The triangle so formed is consequently self-polar and for a given conic there is an unlimited number of such self-polar triangles.

109. The Diagonal Triangle of an Inscribed Quadrangle or of a Circumscribed Quadrilateral is Self-Polar. The following reciprocal theorems arise immediately from the polar properties already developed.

THEOREM. *The three pairs of opposite sides of a complete quadrangle inscribed in a conic (Fig. 59) intersect in the vertices of a triangle which is self-polar relative to the conic.*

In other words, the diagonal points of a complete quadrangle inscribed in a conic are the vertices of a self-polar triangle for that conic.

THEOREM. *The three pairs of opposite vertices of a complete quadrilateral circumscribed to a conic (Fig. 59) determine the sides of a triangle which is self-polar relative to the conic.*

In other words, the diagonals of a complete quadrilateral circumscribed to a conic are the sides of a self-polar triangle for that conic.

110. Pole and Polar are Projectively Related. If PQR is a self-polar triangle with respect to a given conic (Fig. 63),

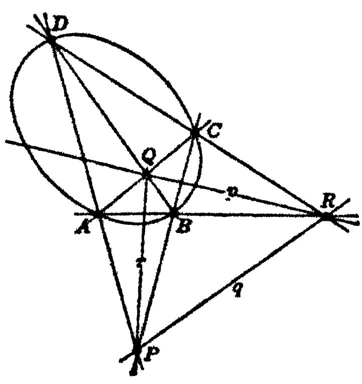


FIG. 63

let us suppose that the points A, C, and Q remain fixed while D moves along the conic. The line CD will then generate a pencil of rays about C projective to the pencil about A, and the range of points R on the line PR will be projective to the range described by P on the same line. Hence QR, the polar of P, will generate a pencil of rays about Q projective to the range of points P. From this we have the following important result.

THEOREM. *If a point P moves on any straight line q, its polar p with respect to a given conic will rotate about a point Q, the pole of q, and will generate a pencil of rays projective to the range of points P; and conversely.*

The pencil of rays described by the rotating polar is projective to the range of points described by the pole.

111. Pole and Polar Relations in a Cone. The pole and polar theory for a plane may be transferred immediately to a cone of the second order by projection. For example, we may make the following statements.

If there are given a cone of the second order and a ray s passing through its vertex, and through s there are passed two planes at random, α and β, cutting the cone, the following rays lie in one plane σ passing through the vertex.

(*a*) The ray in either plane α or β harmonically separated from s by the cone;

(*b*) The common ray of the planes tangent to the cone along the lines of intersection of either α or β with the cone;

(*c*) The rays of contact of the planes through s tangent to the cone, if any;

(*d*) The lines of intersection of opposite faces of the four-edge inscribed in the cone, of which the planes α and β are diagonals.

The plane σ is the polar plane of s with respect to the cone, and s is the pole-ray of σ.

Similarly, the properties of conjugate lines and points may be transferred to the cone, yielding conjugate planes and rays all passing through the vertex.

112. Pole and Polar in a Ruled Quadric Surface. A ruled surface of the second order is projected from any point P, not on the surface, by a pencil of planes enveloping a cone of the second order (§ 82) and the planes of this pencil are tangent to the surface at points of a plane section of the surface. This plane of section, that is, the plane containing the points of contact of planes through P, is called the polar plane of the point P. Any ray through P, meeting the surface and not lying in a tangent plane, is cut harmonically at P, the polar plane, and the surface. In other words, P is harmonically separated by the surface from the points of its polar plane.

EXERCISES

1. By linear constructions determine the polar of a given point and the pole of a given line with respect to a conic of which only five points or five tangents are known.

2. From a given point in its plane draw the tangents to a given conic; that is, draw the two rays of a pencil of the second order which pass through a given point.

3. If four points on a straight line are harmonic, their polars relative to a given conic pass through one point and are likewise harmonic.

4. If a triangle is self-polar with respect to a circle, show that the center of the circle is the orthocenter of the triangle.

5. A triangle ABC, of which a, b, c, are the sides opposite corresponding vertices, is inscribed in a conic and a', b', c' are the tangents to the conic at corresponding vertices, forming a circumscribed triangle $A'B'C'$. The pairs of sides a, a'; b, b'; c, c', intersect in points of a straight line k and the lines AA', BB', CC', pass through one point K such that the line k is the polar of K relative to the conic.

6. A variable tangent to a conic intersects two fixed tangents and their chord of contact in three points which, with the point of contact of the variable tangent, form a harmonic range.

7. Construct a conic through three points, not collinear, with respect to which a given point is the pole of a given line. Is there more than one such conic?

8. Construct a conic through a given point with respect to which two given points are the poles, respectively, of two given lines.

9. Construct the circle with respect to which a given triangle is self-polar.

10. If there are given one point of a conic and a self-polar triangle, show that three other points of the conic may be found.

CHAPTER X

APPLICATIONS OF THE POLE AND POLAR THEORY

113. The Polar Reciprocal of a Conic. In § 110, if the point P should move on a curve of the second order g instead of on a straight line, its polar p with respect to a fixed conic k would generate a pencil of rays of the second order; that is, it would generate the system of tangents to another conic. For, in that case, the path of P is the locus of the intersections of successive pairs of homologous rays in two projectively related pencils S_1 and S_2, generating the conic g. The polar of P relative to k, therefore, joins successive pairs of homologous points in two projective ranges, u_1 and u_2, where u_1 and u_2 are the polars of S_1 and S_2 respectively, relative to the conic k, and the pairs of homologous points on them are the poles of homologous rays in S_1 and S_2. Hence the polar of P generates a pencil of rays of the second order enveloping a third conic h. This relation may be stated in the following theorem.

THEOREM. *If a moving point describes a conic, its polar relative to a fixed conic describes the system of tangents to a third conic.*

The first and third conics, g and h, are mutually reciprocal relative to the fixed conic k; that is to say, the tangents to either of them are the polars of the points of the other.

If the conics g and k have a point in common, the polar of that point relative to k is the tangent to k at the point, and consequently, for every point which the conics g and k have in common, there is a tangent common to h and k. Moreover, for every tangent which g and k have in common there is a point common to h and k.

114. Conjugate Points on Non-Conjugate Lines and Conjugate Lines through Non-Conjugate Points. If two lines u and v are non-conjugate relative to a given conic, that is, neither passes through the pole of the other, to every point of one of these lines there is one and only one conjugate point on the other. For, if a point A lies on the line u, it is conjugate to every point of its polar and to no other points. The polar of A will intersect the line v in a point A', conjugate to A.

If now the point A moves along the line u, its polar will rotate about U, the pole of u, and will generate a pencil of rays projective to the range of points A (§ 110). The range of points A' on the line v is a section of this pencil and it is therefore projective to the range of points A. The lines joining the pairs of conjugate points in u and v will form a pencil of rays of the first or the second order according as the common point of u and v is or is not self-conjugate, that is, according as u and v do or do not intersect on the given conic.

Reciprocally, if two points U and V are non-conjugate with respect to a given conic, that is, neither lies on the polar of the other, to every line a passing through one of these points, U, there is one and only one line a' passing through the other, V, which is conjugate to it. For the line a is conjugate to lines through its pole and to no others, and there is but one of these which passes through V.

The pairs of conjugate lines in U and V form projective pencils of rays which generate a curve of the first or second order according as the line UV is or is not self-conjugate; that is, according as UV is or is not tangent to the given conic.

Briefly it may be stated that the ranges of conjugate points on two non-conjugate lines and the pencils of conjugate rays through two non-conjugate points are projective.

115. Conjugate Points on the Sides of an Inscribed Triangle.

THEOREM. *If a triangle is inscribed in a conic, the lines joining pairs of conjugate points on two of the sides are conjugate to the third side; that is, they pass through the pole of the third side.*

For since no two sides of the inscribed triangle are conjugate relative to the conic, and since the point in which the two chosen sides intersect; namely, one vertex of the triangle, is self-conjugate (Fig. 64), the lines joining pairs of conjugate points in them will pass through one point (§ 114). Two of

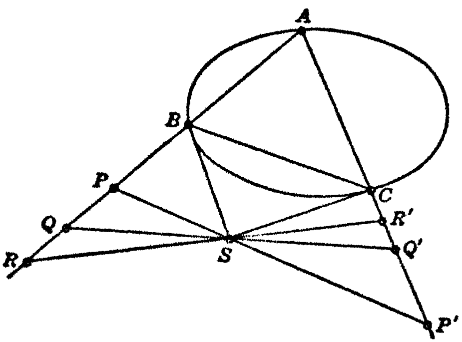


FIG. 64

these lines are the tangents at the other two vertices of the triangle and they intersect in the pole of the third side. Hence all lines joining pairs of conjugate points in two sides will pass through the pole of the third side and, consequently, will be conjugate to the third side.

The converse to this theorem may be stated as follows.

THEOREM. *If a triangle is inscribed in a conic, any line conjugate to one side, relative to the conic, cuts the other two sides in conjugate points.*

Also, the following theorem is converse.

THEOREM. *If a given point is the pole of one side of a triangle, relative to a conic through the opposite vertex, and lines through the pole intersect the other two sides in conjugate points, the triangle is inscribed in the conic.*

The reciprocal theorem takes the following form.

THEOREM. *If a triangle is circumscribed to a conic, the points of intersection of pairs of conjugate lines through two of the vertices are conjugate to the third vertex, that is, they lie on the polar of the third vertex.*

The proof of this theorem follows from the proof of the preceding one by reciprocation. The converse theorem may be stated as follows.

THEOREM. *If a triangle is circumscribed to a conic, the points of the polar of one vertex are projected from the other two vertices by lines which are conjugate relative to the conic.*

116. Conics through Four Points or Touching Four Lines have a Common Self-Polar Triangle. If the points A, B, C, D are the vertices of a complete quadrangle inscribed in a conic (Fig. 59) and the lines a, b, c, d, tangents at these vertices, are the sides of a complete quadrilateral circumscribed to that conic; the diagonal points, P, Q, R, of the inscribed quadrangle are the vertices of a triangle which is self-polar with respect to the conic, and the diagonals of the circumscribed quadrilateral are the sides of the same self-polar triangle (§ 109). Consequently, the self-polar triangle determined by four points of a conic is the same as the self-polar triangle determined by the tangents at those points. Moreover, this triangle is self-polar with respect to any conic through the four points A, B, C, D, and also with respect to any conic to which the four lines, a, b, c, d, are tangent.

Two or more conics, therefore, which intersect in four points A, B, C, D, have a common self-polar triangle whose vertices are the three diagonal points of the complete quad-

rangle A, B, C, D, inscribed to all the conics; and, two or more conics having four tangents, a, b, c, d, in common, have likewise a common self-polar triangle whose sides are the three diagonals of the complete quadrilateral a, b, c, d, circumscribed to all the conics.

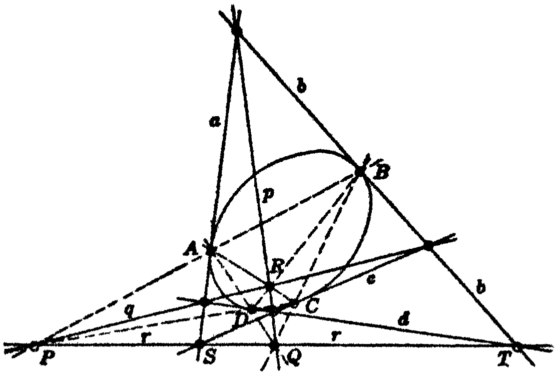


FIG. 59 (*bis*)

117. The Common Self-Polar Triangle of two Conics may be determined either by their Common Points or by their Common Tangents. Two conics may intersect in four points and at the same time may have four tangents in common, and it remains to be seen that the self-polar triangle determined by the four common tangents is the same as that determined by the four points of intersection.

Suppose two given conics k_1 and k_2 intersect in the points A, B, C, D (Fig. 65), and have four tangents, x, y, z, w, in common. The diagonals p, q, r of the complete quadrilateral x, y, z, w, are the sides of a triangle PQR which is self-polar with respect to both conics (§ 109). The vertex P is therefore harmonically separated from any point of its polar by either conic (§ 103). If the secant AP is drawn cutting the polar of P at the point M, the conic k_1 a

second time at C_1, and k_2 at C_2, the sets of points, A, P, C_1, M, and A, P, C_2, M, are both harmonic. Therefore, C_1 and C_2 must coincide, and the secant AP passes through C, a common point of the two conics. Similarly, BP passes through the intersection D of the two conics. That is, AC and BD pass through P; also, AB and CD pass through Q; and AD and BC pass through R.

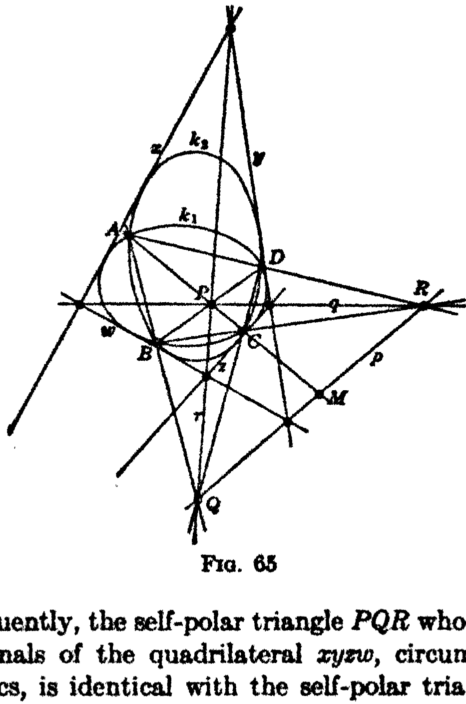


FIG. 65

Consequently, the self-polar triangle PQR whose sides are the diagonals of the quadrilateral $xyzw$, circumscribed to both conics, is identical with the self-polar triangle whose vertices are the diagonal points of the quadrangle $ABCD$, inscribed to both conics.

118. Two Intersecting Conics have only one Self-Polar Triangle in Common. The conics k_1 and k_2 (Fig. 65) intersecting in four points A, B, C, D, can have no triangle

self-polar relative to both of them, other than PQR whose vertices are the diagonal points of the complete quadrangle $ABCD$. For if any point P' different from P, Q, or R is a vertex of a triangle self-polar relative to both conics, its polars relative to k_1 and k_2 must coincide. Of the lines $P'A$ and $P'B$, one or both, say the former, intersects the two conics a second time at different points, C_1 and C_2; the harmonic conjugate of P' relative to A and C_1 lies on the polar of P' relative to k_1, and the harmonic conjugate of P' relative to A and C_2 lies on the polar of P' relative to k_2. Since C_1 and C_2 do not coincide, the harmonic conjugates of P' relative to A and C_1, A and C_2, do not coincide, and therefore the polars of P' relative to the two conics cannot coincide. Consequently, P' cannot be a vertex of a common self-polar triangle for the two conics.

Similarly, we can show that two conics having four common tangents can have but one self-polar triangle in common.

119. Two Quadrangles having the same Diagonal Points may be Inscribed in one Conic.

THEOREM. *If two complete quadrangles $ABCD$ and $A'B'C'D'$ have the same diagonal points P, Q, R, the eight vertices lie on one conic, or else they lie, four by four, on two straight lines.*

Suppose the pairs of sides of the two given quadrangles AB and CD, $A'B'$ and $C'D'$, intersect at P (Fig. 66); AC and BD, $A'C'$ and $B'D'$, intersect at Q; AD and BC, $A'D'$ and $B'C'$, intersect at R.

Then, if any three points, as A, B, A', lie on one straight line, B' must lie on that same line, since AB and $A'B'$ both pass through P. Also, PQ and PR are harmonically separated both by AB and CD, and by $A'B'$ and $C'D'$ (§ 33). Since AB coincides with $A'B'$, CD must coincide with $C'D'$, and the eight vertices lie on two lines through P.

If, however, no three vertices are collinear, a conic may

be passed through any five of them, say through A, B, C, D, A', and for this conic the triangle PQR is self-polar. It remains to show that B', C', D', lie on this conic.

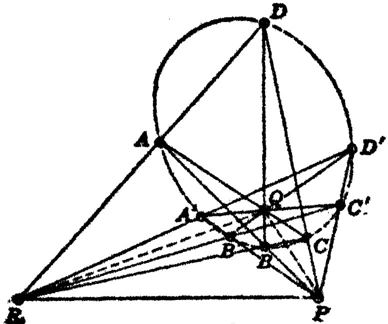


FIG. 66

Join PA' cutting QR at E and cutting the conic a second time at B_1'. Since QR is the polar of P with respect to the conic, P, A', E, B_1', are harmonic points. But the rays $Q(PA'EB')$ are harmonic (§ 33). Hence $PA'EB'$ are harmonic points, and B_1' must coincide with B'. That is, B' lies on the conic determined by A, B, C, D, A'.

Similarly, by joining A' successively with Q and R, we can show that C' and D' lie on the conic; that is, all eight vertices are on the conic determined by any five of them.

The reciprocal theorem, which may be demonstrated in an entirely analogous manner, is as follows.

THEOREM. *If two complete quadrilaterals, abcd and a'b'c'd', have the same diagonals, p, q, r, the eight sides are tangents to one conic, or else they pass, four by four, through two points.*

120. A Quadratic Transformation arising from the Theory of Pole and Polar. As an illustration of the application of the theory of poles and polars to so-called quadratic transformations in a plane the following theorem may be stated.

THEOREM. *If a straight line v and a point U, not the pole of v and not lying on v, are given in the plane of a fixed conic, and*

on each ray through U there is determined that point which is conjugate, relative to the fixed conic, to the intersection of this ray with v, all such conjugate points lie on a conic.

This conic passes through U, through V, the pole of v, and through the points of contact of the tangents, if any, which can be drawn from U to the fixed conic.

If v should pass through one of these points of contact, that is, if UV should be tangent to the fixed conic, the locus is a range of points of the first order instead of a conic.

through each point of v there is drawn that ray which is conjugate, relative to the fixed conic, to the line joining this point with U, all such conjugate rays are tangent to a conic.

This conic touches the line v, also the line u, the polar of U, and the tangents at the two points of intersection, if any, of the given line v with the fixed conic.

If U lies on one of these tangents, that is, if u and v intersect on the fixed conic, the locus is a pencil of rays of the first order instead of a system of tangents to a conic.

In the theorem on the left, if any ray a through the given point U (Fig. 67) intersects the given line v at a point A, the polar of A relative to the given conic is a ray a' through V the pole of v. This ray a' intersects a at a point A', conjugate to A, and it is the locus of A' to which the theorem refers.

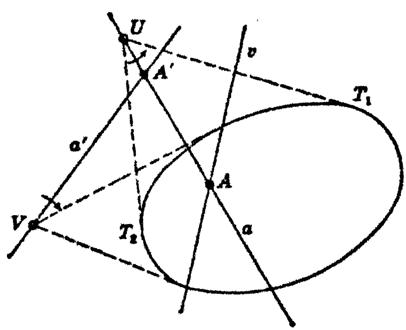


FIG. 67

The range of points A on the line v is projective to the pencil of polars through V (§ 110), and the locus of A' is the conic generated by the projective pencils of rays U and V.

If T_1 and T_2 are the points of contact of rays through U

tangent to the given conic, they are conjugate to the points in which these tangents intersect v and hence are points of the locus. The locus of conjugate points is then a conic through the two centers U and V and through the points of contact T_1 and T_2 of tangents from U to the given conic.

In case the line v passes through T_1 or T_2, the pole V lies on one or the other of the tangents from V and the line UV in the two pencils is self-corresponding. The locus in this case is a straight line, or a range of points of the first order.

The theorem on the right is the reciprocal of that on the left and is proved analogously.

In the theorem on the left, again, to any line v of the plane there corresponds a particular conic passing through the fixed points U, T_1, and T_2, the points of the line corresponding, one to one, to the points of the conic. To all lines of the plane, there corresponds, therefore, the system of conics passing through three fixed points.

Similarly, in the theorem on the right, to the lines through any point U of the plane there corresponds the system of tangents to a conic, and all such conics whose tangents correspond to rays through points U of the plane have three tangents in common; namely, the line v and the tangents to the fixed conic at the points where v cuts it.

121. The Locus of Mid-Points of Chords through a Fixed Point. A special case of the theorem on the left (§ 120) is presented in the following theorem.

THEOREM. *The mid-points of the chords of a given conic drawn through a fixed finite point of the plane lie on a second conic.*

The line v in this case is the infinitely distant line of the plane and the conjugate point of the intersection with v of any chord through the fixed point U is the mid-point of that chord, since it is harmonically separated from the infinitely distant point of the chord by the given conic.

122. A Second Quadratic Transformation. The following considerations present a second illustration of quadratic transformations arising from the polar theory.

Given two conics in a plane, any point A of the plane has a polar with respect to each conic. If these polars intersect in A', the points A and A' are conjugate with respect to both conics. The polars of A', likewise, intersect in A, and the relation between the two points A and A' is mutual. Similarly, any straight line of the plane may be correlated to another line to which it is conjugate with respect to both conics.

If the two given conics are k_1 and k_2 and the point A moves along a fixed line m, the polar of A relative to the conic k_1 will rotate about M_1, the pole of m with respect to that conic, and the polar of A relative to the conic k_2 will rotate about M_2, the pole of m with respect to that conic. The pencil of rays M_1 is projective to the pencil M_2, each being projective to the range of points m. Hence the two pencils, in general, will generate a conic which is the locus of A' and which passes through the poles M_1 and M_2.

For the point in which the line m intersects a side p of the common self-polar triangle of the two conics, the homologous rays in M_1 and M_2 will both pass through the opposite vertex P of the self-polar triangle, and similarly for the other two sides. Hence the conic generated by M_1 and M_2 passes through the three vertices of the self-polar triangle. In such a correlation, then, we have the following relations.

The points of a straight line are correlated, in general, to the points of a conic which passes through the vertices of the common self-polar triangle of the two given conics.	The lines through any point of the plane are correlated, in general, to the rays of a pencil of the second order to which belong the sides of the common self-polar triangle of the two given conics.
The lines of the plane, therefore, are correlated, in general,	The lines through all points of the plane, therefore, are cor-

to the conics through three fixed points.

related in general to the pencils of rays of the second order enveloping conics which touch three fixed lines.

If the line m passes through a vertex P of the self-polar triangle, the poles M_1 and M_2 will lie on the opposite side p of the self-polar triangle, and the two pencils M_1 and M_2 in that case will generate a range of points of the first order, since in them the common ray p is self-corresponding. In this particular case, for the point in which the line m intersects the side p of the self-polar triangle, both polars pass through the vertex P, which is therefore a point of the resulting range.

The polars of a vertex of the self-polar triangle, relative to the two given conics, coincide in the opposite side of that triangle, and hence any point of a side of the self-polar triangle is correlated to the opposite vertex. Similarly, any line through a vertex of the self-polar triangle is correlated to the opposite side of that triangle.

To the points of a straight line passing through a vertex P of the self-polar triangle will correspond the points of the opposite side of that triangle and of another straight line through the same vertex P. In this case, the conic corresponding to the line may be said to degenerate into two straight lines; namely, a side of the self-polar triangle and a line through the opposite vertex.

Similarly, to the lines through any point of a side of the self-polar triangle will correspond the lines through the opposite vertex and the lines through another point of that side. In this case, the pencil of rays of the second order degenerates into two pencils of rays of the first order; namely, the pencil whose center is the opposite vertex of the self-polar triangle and another whose center lies on the chosen side.

EXERCISES

1. If two tangents to a given conic vary so that their chord of contact envelops a second conic, their point of intersection will trace a third conic.

2. Two triangles which are self-polar with respect to the same conic are inscribed in a second conic and are circumscribed to a third conic.

SUGGESTION. If $P_1Q_1R_1$ and $P_2Q_2R_2$ are the given triangles, it follows (§ 114) that $P_1(Q_1R_1Q_2R_2) \barwedge P_2(R_1Q_1R_2Q_2)$. Also (§ 60), $P_2(R_1Q_1R_2Q_2) \barwedge P_2(Q_1R_1Q_2R_2)$. Hence, $P_1(Q_1R_1Q_2R_2) \barwedge P_2(Q_1R_1Q_2R_2)$ and the six vertices lie on a conic. Reciprocally, the six sides are tangents to a conic.

3. If PQR is a self-polar triangle for a given conic, and ABC is a triangle inscribed in that conic, such that two of its sides, AB and AC, pass through R and Q, respectively, the third side will pass through P, and each side of the inscribed triangle is cut harmonically by the conic, the vertex of the self-polar triangle through which it passes, and the opposite side of that triangle. Moreover, the triangles ABC and PQR are perspective, the center being a point of the conic.

4. If two conics intersect in four points, the eight tangents at those points are tangent to one conic, or else they pass, four and four, through two points; and, reciprocally, if two conics have four tangents in common, the eight points of contact lie on a conic or else they lie, four by four, on two straight lines.

SUGGESTION. The tangents form two quadrilaterals (§ 119) which have the same diagonals; namely, the sides of the diagonal triangle of the complete quadrangle whose vertices are the points of intersection of the two conics.

5. If two conics intersect in only two actual points and have two tangents in common, show that of the common self-polar triangle one vertex and the opposite side can be constructed.

6. Construct the self-polar triangle common to two conics which have four actual tangents in common but no actual points in common.

7. All conics for which a given triangle PQR is self-polar and which pass through a given point A, pass also through three other fixed points B, C, D (Cremona § 268).

8. Construct a conic through two given points for which a given triangle is self-polar (Cremona § 269).

CHAPTER XI

DIAMETERS AND AXES—ALGEBRAIC EQUATIONS OF CONICS

123. The Diameter of a Conic Defined. In the general theory of poles and polars it was found that if, through any point in the plane of a conic, secants are drawn to the conic, and the harmonic conjugate of the point, relative to the curve, is found on each secant, all those harmonic conjugate points lie on a straight line, the polar of the point with respect to the curve (§ 103).

If the chosen point is infinitely distant (Fig. 68), the secants through it are parallel and the harmonic conjugate points become the mid-points of the parallel chords. The polar in this case is called a ***diameter*** of the conic.

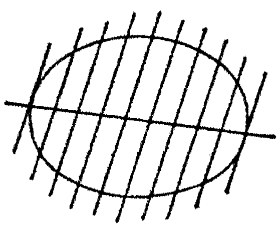


Fig. 68

Definition. A diameter of a conic is the polar of an infinitely distant point relative to the conic.

A diameter bisects a system of parallel chords of the conic, each of which is conjugate to the diameter since it passes through the infinitely distant pole of the diameter.

124. Extremities of a Diameter. By definition, a diameter is the unlimited polar of an infinitely distant point, but it is sometimes convenient to speak of it as a chord of the conic, in which case its ***extremities*** are the points of intersection with the curve and its length is the measure of the line-segment intercepted between the extremities.

Since the polar of a point relative to a conic passes

through the points of contact of the tangents drawn from the point (§ 101) the tangents at the extremities of a diameter are parallel to each other and to the chords bisected by that diameter. The tangents at the extremities of any chord of a conic intersect on the diameter to which the chord is conjugate.

125. The Center of a Conic. Any two diameters of a conic intersect in the pole of the infinitely distant line since each diameter is the polar of an infinitely distant point with respect to the given conic. This intersection is called the *center* of the conic.

Definition. The center of a conic is the pole of the infinitely distant line with respect to the conic.

All diameters of the conic pass through the center, since their poles lie on the infinitely distant line, and any line through the center is a diameter since its pole is infinitely distant. All chords through the center of a conic are bisected at the center.

126. Conjugate Diameters. Among the chords conjugate to any diameter and bisected by it, one is a diameter. To any diameter, therefore, there is a conjugate diameter.

The tangents at the extremities of any diameter are parallel to the conjugate diameter.

If two diameters are conjugate, each bisects chords parallel to the other and each passes through the infinitely distant pole of the other. The diameters and their conjugate diameters form two projectively related pencils of rays with the same center.

Any two conjugate diameters of an ellipse or hyperbola form, with the infinitely distant line of the plane, a triangle which is self-polar with respect to the conic.

Every chord parallel to one of two conjugate diameters is bisected by the other. Hence to construct the diameter

conjugate to any given diameter it is necessary only to draw the line through the mid-points of two chords parallel to that diameter.

127. Diameters of a Parabola. If the given conic is a parabola, the infinitely distant line of the plane is tangent to the curve (§ 72) and its pole is the infinitely distant point of contact. All diameters pass through the point of contact of the infinitely distant line and are therefore parallel (Fig. 69). The center of a parabola lies on the curve and is infinitely distant.

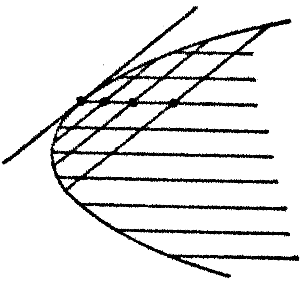


Fig. 69

A diameter of a parabola has only one accessible extremity. The tangent at this extremity is parallel to the chords conjugate to the diameter. To a diameter of a parabola there is no conjugate diameter.

128. Diameters of an Ellipse or a Hyperbola. If the given conic is an ellipse, the infinitely distant line of the plane lies wholly outside the curve (§ 72) and the center of the ellipse, consequently, lies inside the curve. Every diameter of an ellipse intersects the curve at two points, and to every diameter there is a conjugate diameter which likewise intersects the curve.

If the conic is a hyperbola, the infinitely distant line cuts it at two points (§ 72) and the center lies outside the curve. To each diameter there is a conjugate diameter, and of these one intersects the curve while the other does not (§ 107).

The asymptotes of a hyperbola (§ 72) intersect at the center and are harmonically separated by all pairs of conjugate diameters (§ 107).

129. Chords which Bisect each other are Diameters. If two chords of a central conic (an ellipse or a hyperbola) bisect each other, their point of intersection is the center of the curve. For the harmonic conjugate of this point on either chord, with respect to the curve, is infinitely distant and the polar of the point is the infinitely distant line of the plane. The point, therefore, is the center of the curve. From this it follows that two chords of a parabola cannot bisect each other.

130. Conjugate Diameters in Relation to Inscribed and Circumscribed Parallelograms. Since the diagonals of any parallelogram bisect each other, the diagonals of a parallelogram inscribed in a central conic (Fig. 70) are diameters, their point of intersection being the pole of the infinitely distant line and consequently the center of the conic.

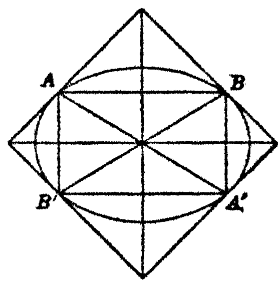


Fig. 70

Conversely, if A and A' are two points of a conic collinear with the center, and likewise B and B', the figure $ABA'B'$ is an inscribed parallelogram since the pairs of opposite sides of the quadrangle so formed intersect on the polar of the center.

Moreover, the tangents at A and A', B and B', also intersect on the polar of the center and are pairs of opposite sides of a circumscribed parallelogram whose diagonals pass through the center and are diameters parallel to the sides of the inscribed parallelogram (§ 93).

Each of these diagonals (diameters) bisects chords of the conic parallel to the other; that is, they bisect the sides of the inscribed parallelogram and are therefore conjugate diameters. These relations are stated in the following theorem.

THEOREM. *The diagonals of a parallelogram circumscribed to a conic are conjugate diameters of the conic and the sides of a parallelogram inscribed to a conic are parallel to a pair of conjugate diameters.*

131. Chords Parallel to Conjugate Diameters. If any point of a central conic is joined to the extremities of a diameter not passing through it, the chords so drawn are parallel to a pair of conjugate diameters. For the diameter parallel to either of these chords bisects the other chord, and of the two diameters so drawn each bisects chords parallel to the other.

If the parallelogram is completed, of which these two, chords are adjacent sides, the fourth vertex lies on the curve at the extremity of the diameter drawn through the given point.

From this it follows that having given one point of a central conic and a pair of conjugate diameters in position, though not in length, three other points of the curve may be found, and if two pairs of conjugate diameters are given in position, five other points of the curve may be found and hence the curve can be constructed.

132. Axes of a Conic. DEFINITION. Conjugate diameters of a conic which are at right angles to each other are called the *axes* of the curve and their extremities are the *vertices* of the curve.

An axis of a conic therefore bisects the chords perpendicular to it.

A parabola has only one axis, namely, that diameter which bisects the chords perpendicular to the common direction of all diameters.

133. Only a Circle has more than one Pair of Axes.

THEOREM. *If a conic has more than one pair of axes, it must be a circle.*

For if a given conic has two pairs of conjugate diameters at right angles (Fig. 71), and if A is any point of the conic, A' being the other extremity of the diameter through A, two rectangles can be constructed on the diameter AA' with sides parallel, respectively, to the pairs of conjugate diameters. These rectangles will be inscribed in the conic (§ 131) and will determine six points of it. But a circle can be described through these same six points. Hence the circle and the given conic must coincide (§ 67).

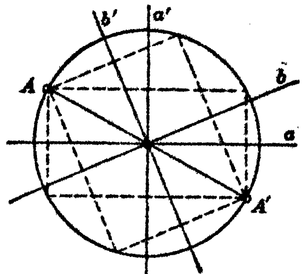


FIG. 71

134. Construction of the Axes. To determine the axes of an ellipse or hyperbola, let AA' be any diameter of the given conic and on it as diameter describe a circle (Fig. 72), which in general will cut the given conic at A and A', and at two other points, B and B'. The chords BA and BA' are therefore parallel to a pair of conjugate diameters of the curve (§ 131) and they are at right angles to each other since they lie in a semicircle. Hence the diameters parallel to these chords are axes of the given conic.

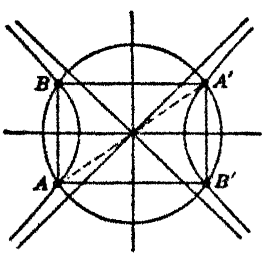


FIG. 72

It is only in case the tangents to the conic at the extremities of AA' are perpendicular to that diameter that the circle will not cut the given conic. But in that case AA' is itself an axis.

An ellipse is cut by both its axes; hence it has four vertices. A hyperbola has but two vertices and a parabola has but one actual vertex.

Since an axis bisects the chords perpendicular to it, each axis divides the curve into two symmetrical parts.

The axes of a hyperbola and the asymptotes are a set of harmonic rays (§ 107) of which one pair are at right angles. Hence the axes of a hyperbola bisect the angles between the asymptotes (§ 38).

135. Rectangular Hyperbolas. Definition. A *rectangular hyperbola* is one whose asymptotes are at right angles.

Any pair of conjugate diameters and the asymptotes of a hyperbola form a set of harmonic rays (§ 107), and the angles between two conjugate diameters of a rectangular hyperbola are, consequently, bisected by the asymptotes (§ 38). Hence, if one of two conjugate diameters of a rectangular hyperbola rotates about the center in either sense, the other diameter will rotate about the center in the opposite sense. The two pencils of rays formed by these rotating diameters are equiangular and oppositely projective, the pairs of corresponding rays always making equal angles with either asymptote.

Since the lines joining any point of a central conic to the extremities of a diameter are parallel to a pair of conjugate diameters (§ 131), a point P of a rectangular hyperbola (Fig. 73) is projected from the extremities of a diameter AA' by rays making equal angles with either asymptote. As P moves along the hyperbola, the rays AP and $A'P$ will rotate about A and A', respectively, in opposite senses, generating pencils of rays in which the angle between any two rays of the one pencil is equal to the angle between the homologous rays of the other. Hence we have the following theorem.

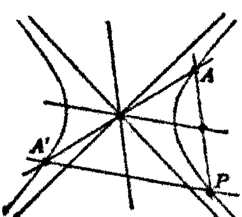


Fig. 73

THEOREM. *A rectangular hyperbola is projected from the extremities of any diameter by equal and oppositely projective pencils of rays.*

Conversely, two equal and oppositely projective pencils of rays in the same plane, which are not perspective, generate a rectangular hyperbola of which the line joining the centers of the pencils is a diameter. The two pairs of homologous rays in the two pencils which are parallel (§ 72) have the directions of the asymptotes of this hyperbola.

It may be of interest to note that two equal and directly projective pencils of rays in the same plane generate a circle. If pairs of homologous rays in the two pencils are at right angles, the centers of the generating pencils are the extremities of a diameter.

136. Two particular Pencils of Rays generating a Rectangular Hyperbola.

THEOREM. *The lines drawn through a given point in the plane of a central conic perpendicular to the diameters of that conic, intersect the conjugate diameters in points of a rectangular hyperbola.*

For the conjugate diameters of the given conic k form a pencil of rays projective to the diameters (§ 126) and hence projective to the rays through the given point S perpendicular to those diameters. In these two pencils there are two pairs of homologous rays parallel to each other; namely, the ray through S perpendicular to either axis of k and its homologous ray, the other axis.

The locus of intersections is therefore a hyperbola k_1, and since the pairs of homologous rays determining the infinitely distant points of k_1 have the directions of the axes of k, the asymptotes of k_1 are at right angles to each other and the hyperbola is therefore rectangular. This hyperbola passes through S and through the center of k.

137. Normals to a Conic. DEFINITION. The *normal* to a curve at a given point is the line through that point perpendicular to the tangent at the point.

A normal to a conic at a given point P is normal therefore to the diameter conjugate to CP where C is the center of the conic.

In the two conics k and k_1 (§ 136), of which the latter is a rectangular hyperbola dependent on the position of a given point S, if P is a point of intersection of these two conics, the line SP is normal to the conic k, for SP is normal to the diameter of k conjugate to the diameter through P. On the other hand, if Q is any point of k not common to k and k_1, the line SQ cannot be normal to k since it is not normal to the diameter conjugate to CQ, where C is the center of k. Since k and k_1 can have at most four points in common, there are at most four normals to k which pass through the point S; that is, there are at most four normals to any conic which pass through an arbitrary point. This property was known to Apollonius.

138. The Conjugate Normals to Rays through a Point Envelop a Parabola. In the plane of a given conic, a point S is chosen, not on an axis, and to each ray through S there is drawn that ray through its pole, with respect to the given conic, which is normal to it; that is, its conjugate normal is drawn. These conjugate normal rays envelop a parabola of which the polar of S and the axes of the given conic are tangents.

If through any point S_1, not on the polar of S, rays are drawn perpendicular to the rays of S, they constitute a pencil of rays projective to the pencil S, and hence projective to the range of poles on the polar of S (§ 110). The conjugate normals join the poles of the rays of S to corresponding points of the range on the infinitely distant line determined by the pencil S_1. They constitute, therefore, a pencil of rays

of the second order to which the polar of S and the infinitely distant line belong. The pencil of conjugate normals, therefore, envelops a parabola. Moreover, to the ray of S parallel to either axis of the given conic, the other axis is the conjugate normal; hence, the axes of the given conic are rays of the pencil of the second order; that is, they are tangents to the parabola enveloped by the pencil.

Any tangent to the parabola is conjugate to its normal ray through S, and if a tangent to the parabola is tangent also to the given conic, the conjugate normal through S must pass through the point of contact on the given conic since a tangent to the given conic is conjugate only to rays through its point of contact. The conjugate normal through S to a tangent common to the parabola and the given conic is, therefore, normal to the given conic, and since there are at most four tangents common to the two conics, there are, again, at most four normals to the given conic passing through the arbitrary point S.

139. Construction of a Conic by use of the Ruler only. Let S_1 and S_2 be the extremities of a diameter of a central conic (Fig. 74), and u_1 and u_2 be straight lines in the plane of the conic, parallel to a pair of conjugate diameters. If the points of the conic are projected from S_1 and S_2 on u_1 and u_2, respectively, the ranges of points u_1 and u_2 are similarly projective, that is, the infinitely distant point of u_1 corresponds to the infinitely distant point of u_2. For if P is a variable point of the conic, when S_1P is parallel to u_1, S_2P is parallel to u_2 (§ 131). Hence homologous segments of u_1 and u_2 are proportional (§ 74).

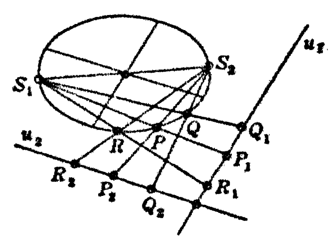


FIG. 74

From this property there is derived a simple construction for an ellipse or a hyperbola by use of the ruler only.

Thus, if u_1 and u_2 are two straight lines on which proportional segments, $A_1, B_1, C_1, D_1, \cdots$ and $A_2, B_2, C_2, D_2, \cdots$ are marked out, and the points $A_1, B_1, C_1, D_1, \cdots$ and $A_2, B_2, C_2, D_2, \cdots$ are projected from centers S_1 and S_2, respectively, so chosen that the line S_1S_2 is not parallel to either u_1 or u_2, the intersections of pairs of homologous rays will lie on a conic of which S_1S_2 is a diameter and for which u_1 and u_2 are parallel to a pair of conjugate diameters.

If the center S_2 is chosen as the infinitely distant point of u_1, that is, if the lines through $A_2, B_2, C_2, \cdots$ are drawn parallel to u_1, the intersections of pairs of homologous rays will lie on a parabola through S_1 of which u_1 is a diameter.

140. The Segments of a Line Intercepted between a Hyperbola and its Asymptotes are Equal. Any chord of a hyperbola parallel to one of two conjugate diameters is bisected by the other (§ 126). Also, since conjugate diameters of a hyperbola are harmonically separated by the asymptotes (§ 128), the segment of a line parallel to one of two conjugate diameters and intercepted between the asymptotes is bisected by the other diameter.

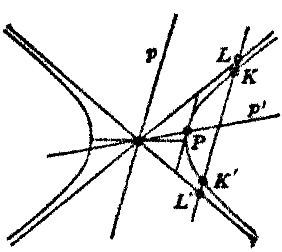


FIG. 75

Hence, if KK' (Fig. 75) is a chord of a hyperbola parallel to a diameter p, and LL' is the segment of the same line intercepted between the asymptotes of the hyperbola, both KK' and LL' are bisected by the conjugate diameter p'. Therefore the segments KL and $K'L'$ are equal; and if LL' is tangent to the hyperbola at P, the segments LP and $L'P$ are equal. Hence we have the following theorem.

THEOREM. *On any straight line cutting a hyperbola and its asymptotes, the two segments intercepted between the asymptotes and the curve are equal; and on any line tangent to a hyperbola, the segment intercepted between the asymptotes is bisected at the point of contact.*

This theorem furnishes a very simple construction for a hyperbola of which one finite point and the asymptotes are given. For if any secant is drawn through the given point, it intersects the asymptotes in two points which are equally distant from the points of the curve lying on the secant.

141. The Triangle formed by the Asymptotes of a Hyperbola and a Variable Tangent. From Brianchon's theorem, it is known that the diagonals of a simple quadrilateral circumscribed to a conic and the lines joining the points of contact in pairs of opposite sides pass through one point (§ 93).

Suppose the given conic is a hyperbola (Fig. 76), and the asymptotes, intersecting at O, the center of the curve, are one pair of opposite sides of a circumscribed quadrilateral, while AB and $A'B'$, tangents to the curve at P and P', respectively, are the other pair of opposite sides. Then AA', BB', and PP' must be parallel, since they have a point in common with the infinitely distant line joining the points of contact of the asymptotes.

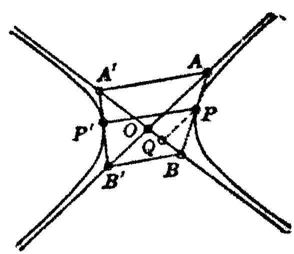


FIG. 76

The triangles having a common base BB' and vertices at A and A', respectively, are equal in area. From these take away the common triangle BOB' and the remaining areas AOB and $A'OB'$ are equal. Since AB and $A'B'$ are any tangents whatsoever, the following theorem may be stated.

THEOREM. *The triangle formed by the asymptotes of a hyperbola and a variable tangent is of constant area.*

Moreover, since AOA' and $B'OB$ are equiangular triangles, we have the relation

$$OA \cdot OB = OA' \cdot OB'.$$

In other words, the product of the segments of the asymptotes of a hyperbola cut off by a variable tangent is constant.

142. Equation of a Hyperbola Referred to its Asymptotes as Axes. From the property of § 141 the algebraic equation of a hyperbola, referred to its asymptotes as axes, is readily found.

The point P (Fig. 76) may be any finite point of the hyperbola since it is the point of contact of any tangent AB. From P draw PQ parallel to one asymptote to meet the other asymptote at Q. Then, since P is the mid-point of AB (§ 140), we may write

$$QP = \tfrac{1}{2}\,OA,$$

and

$$OQ = \tfrac{1}{2}\,OB.$$

Designating OQ by x and QP by y, we have by multiplication the relation

$$xy = \tfrac{1}{4}\,OA \cdot OB.$$

Since $OA \cdot OB$ is constant for any given hyperbola (§ 141), the algebraic equation of a hyperbola referred to its asymptotes as axes takes the form

$$xy = \text{a constant.}$$

143. Equation of a Central Conic Referred to a Pair of Conjugate Diameters. In analytic geometry, the algebraic equations of the ellipse and hyperbola are commonly deduced by referring the conics to a pair of conjugate diameters as coordinate axes.

If conjugate diameters, OX and OY, of an ellipse or a hyperbola are chosen as coördinate axes, at least one of them, OX say, must cut the curve at points A and A'. Tangents at these points are parallel to each other and to the conjugate diameter, OY.

Let P be any point of the conic (Fig. 77), and through P draw the tangent to the curve cutting the tangents at A and A' in B and B', respectively, and let any fourth tangent cut the tangents at A and A' in D and D', respectively. Then we have the relation

$$AB \cdot A'B' = AD \cdot A'D'.$$

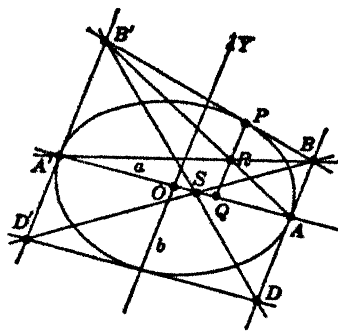


FIG. 77

For, first, $BB'D'D$ is a quadrilateral circumscribed to the conic, and hence BD' and $B'D$ intersect on the line AA' joining the points of contact in a pair of opposite sides, say at S. Then from the similar triangles ASB and $A'SD'$, we have the relation

$$\frac{AB}{A'D'} = \frac{AS}{A'S}.$$

Also, from the similar triangles, ASD and $A'SB'$, we have the relation

$$\frac{AD}{A'B'} = \frac{AS}{A'S}.$$

Therefore

$$\frac{AB}{A'D'} = \frac{AD}{A'B'},$$

or

$$AB \cdot A'B' = AD \cdot A'D'.$$

If the curve is an ellipse and the tangent DD' is drawn parallel to the diameter AA', it is readily seen that the product $AD\cdot A'D'$ is equal to the square of half the diameter conjugate to AA'. Representing the length of this semi-conjugate diameter by b, we have the relation

$$AB\cdot A'B' = b^2 = AD\cdot A'D'.$$

If the curve is a hyperbola (Fig. 78), the segments AB and $A'B'$, also AD and $A'D'$, are measured in opposite directions from A and A' and their product should be considered negative. The tangent DD' may take the position of an asymptote, in which case we have the relation

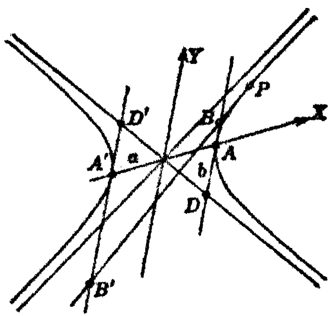


FIG. 78

$$AB\cdot A'B' = AD\cdot A'D' = -b^2,$$

where b is the length of the segment on the tangent at A (or A') intercepted between A (or A') and an asymptote. This value b is defined to be the length of the semi-conjugate diameter in a hyperbola.

For either conic, the tangents at A, A', and P form a circumscribed triangle. Hence, the lines joining the points of contact of the sides to the opposite vertices meet in one point (§ 89); that is to say, AB', $A'B$, and the line through P parallel to AB meet in a point R (Fig. 77). Let PR intersect AA' at Q.

From the simple quadrangle $ABA'B'$ in which one pair of opposite sides intersect at R, the other pair at infinity, one diagonal passes through Q and the other through P, the points Q, R, P, ∞, are harmonic and hence R is the midpoint of QP, a relation which may be proved also from similar triangles.

Let the coördinates of the arbitrary point P of the conic, referred to the conjugate diameters OX and OY as axes, be x and y, in which case $OQ = x$, $QP = y$, and $QR = y/2$. If, then, the semi-diameter OA is denoted by a, we have

$$A'Q = a + x$$

and

$$QA = a - x.$$

Therefore

$$A'Q \cdot QA = a^2 - x^2.$$

From the similar triangles $A'QR$ and $A'AB$ we have the relation

$$\frac{A'Q}{QR} = \frac{A'A}{AB},$$

and from the similar triangles QRA and $A'B'A$ we have the relation

$$\frac{QA}{QR} = \frac{A'A}{A'B'}.$$

Combining these results by multiplication, we find that

$$\frac{A'Q \cdot QA}{QR^2} = \frac{A'A^2}{AB \cdot A'B'}.$$

Substituting the values indicated above and remembering that $AB \cdot A'B'$ equals $+b^2$ or $-b^2$ according as the conic is an ellipse or a hyperbola, we obtain the relation

$$\frac{a^2 - x^2}{y^2} = \pm \frac{a^2}{b^2},$$

which reduces readily to the form

$$\frac{x_2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

This is, therefore, the equation of an ellipse or a hyperbola, the plus sign being used for the former and the minus sign for the latter, a and b being the lengths of the semi-conjugate diameters which were chosen as coördinate axes.

If the conic under consideration is a circle or a rectangular hyperbola, a and b are equal and the equation reduces to the form

$$x^2 \pm y^2 = a^2.$$

Since pairs of conjugate diameters in a rectangular hyperbola are equal, the curve is frequently called an ***equilateral hyperbola.***

144. Equation of a Parabola. In deducing the algebraic equation of a parabola, it is customary to choose as coördinate axes a diameter of the curve and the tangent at its extremity.

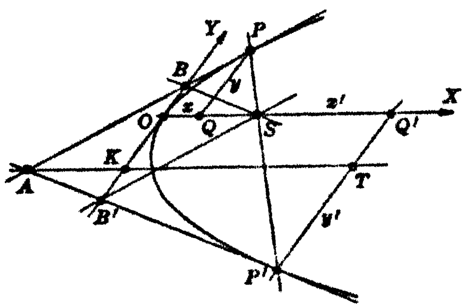


FIG. 79

Let OX (Fig. 79) be any diameter of a parabola and OY be the tangent at its extremity. Let P and P' be any points of the parabola for which the coördinates measured along the diameter OX and parallel to the tangent OY are (x, y) and (x', y'), respectively.

Let PQ and $P'Q'$ parallel to OY meet OX at Q and Q'. Draw the tangents at P and P' intersecting at A and meeting OY at B and B'.

The tangents at the points, P, O, P', and the infinitely distant point of the curve, form a circumscribed quadrilateral of which the diagonals are a line through B parallel to the tangent at P' and a line through B' parallel to the

tangent at P. These diagonals and the lines joining the points of contact on opposite sides, namely, the line PP' and the diameter through O, intersect in one point S (§ 89).

The quadrangle $ABSB'$ is therefore a parallelogram, and the triangle BPS is similar to both the triangles APP' and $B'SP'$. Also, the triangles PQS and $P'Q'S$ are similar. Hence we have the relations

$$\frac{y}{y'} = \frac{PQ}{P'Q'} = \frac{PS}{P'S} = \frac{BP}{B'S} = \frac{BP}{AB},$$

and

$$\frac{y}{y'} = \frac{PQ}{P'Q'} = \frac{PS}{P'S} = \frac{BS}{B'P'} = \frac{AB'}{B'P'}.$$

If the diameter through A is drawn, meeting BB' at K and $P'Q'$ at T, $KT = OQ'$, and from similar triangles, we have the relation

$$\frac{AK}{OQ'} = \frac{AB'}{B'P'} = \frac{y}{y'}.$$

Also, from similar triangles, we have

$$\frac{OQ}{AK} = \frac{BP}{AB} = \frac{y}{y'}.$$

Combining those results by multiplication, we obtain the relation

$$\frac{OQ}{OQ'} = \frac{y^2}{y'^2} = \frac{x}{x'}.$$

This equation is usually written in the form

$$y^2 = 2px,$$

where

$$2p = y'^2/x',$$

a constant for any given parabola.

145. Tangents to a Parabola are Cut Proportionally by Other Tangents. From the process of § 144, it appears incidentally that

$$\frac{BP}{AB} = \frac{AB'}{B'P'},$$

since each of these fractions equals y/y'. That is to say, two tangents of a parabola are cut proportionally by a third tangent, and since the third tangent was chosen arbitrarily we have the property that any two tangents of a parabola are cut proportionally by the remaining tangents, a property which was deduced otherwise in § 74.

EXERCISES

1. Show that the line joining the pole of any chord of a parabola to the mid-point of the chord is a diameter of the parabola and the segment between the pole and the chord is bisected by the curve (Apollonius).

2. If a line is drawn through a given point of the plane, parallel to an asymptote of a hyperbola, the segment of the line intercepted between the point and its polar with respect to the hyperbola is bisected by the curve.

3. Prove that if the points of contact of the sides of a parallelogram circumscribed to an ellipse or hyperbola are joined in order, the inscribed figure is also a parallelogram.

4. Find the center of a conic of which only five points or five tangents are given.

5. Draw the chord of a given conic which is bisected at a given point within it.

6. Prove that the chords of a given conic which are bisected by a given chord envelop a parabola.

7. Given three points of a parabola and the direction if its diameters, construct the parabola and find its axis.

8. If k is a given conic, l and m reciprocal conics in the plane, relative to k, show that m is a hyperbola, a parabola, or an ellipse according as the center of k lies outside, on, or inside, the conic l.

9. If A_1A_2 and B_1B_2 are conjugate diameters of a conic and Q is any point of the conic, the rays A_1Q and A_2Q are intersected by any line parallel to B_1B_2 in conjugate points.

10. If A_1A_2 is a diameter of a conic of which Q is any point, the diameters parallel to A_1Q and A_2Q are conjugate. If these diameters intersect the tangents at A_1 and A_2 in K_1 and K_2, respectively, the line K_1K_2 is tangent to the conic.

11. Parallel lines u_1 and u_2 are drawn through fixed points A_1 and A_2, and a third line u intersects them in points B_1 and B_2. If the line u moves parallel to itself, the pencils of rays A_1B_2 and A_2B_1 generate a hyperbola of which A_1A_2 is a diameter and B_1B_2 has the direction of the conjugate diameter. If $A_1A_2 = 2a$ and $B_1B_2 = 2b$, the equation of the hyperbola referred to these conjugate diameters as axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

12. If two triangles are inscribed in a conic, their six sides are tangents to a second conic; and conversely, if two triangles are circumscribed to a conic, their six vertices lie on a second conic.

SUGGESTION. Given the triangles $A_1B_1C_1$ and $A_2B_2C_2$ inscribed in a conic k_1 and let the sides B_1C_1 and B_2C_2, or a_1 and a_2, intersect the remaining sides b_2, c_2 and b_1, c_1, respectively of the other triangle, in the points B_2', C_2' and B_1', C_1'. Projecting the sets of points on these two sides from A_1 and A_2 we have $A_1(C_1'B_1'B_2C_2) \;\overline{\wedge}\; A_2(B_1C_1C_2'B_2')$ or $C_1'B_1'B_2C_2 \;\overline{\wedge}\; B_1C_1C_2'B_2'$. Joining pairs of homologous points in these two ranges we have four sides of the given triangles tangent to a conic k_2 to which the sides a_1 and a_2 are also tangent.

The converse theorem is proved analogously.

Suppose, now, in the direct theorem, a third triangle $A_3B_3C_3$ is inscribed in the conic k_1 of which the sides b_3 and c_3 are tangent to the conic k_2. The third side a_3 is also tangent to k_2 since the sides a_1, b_1, c_1 and a_3, b_3, c_3 are all tangent to some conic and five of them are tangent to k_2.

Hence the following theorem:

THEOREM. If two conics are so situated that a triangle can be inscribed in one of them which is circumscribed to the other, there is an unlimited number of such triangles (Poncelet, *Proj. Prop.* § 565).

CHAPTER XII

PROJECTIVELY RELATED FORMS OF THE SECOND ORDER

146. Harmonic Relations in Forms of the Second Order. Four points of a conic are defined to be harmonic when they are projected from a fifth point of the conic by harmonic rays, and four tangents to a conic are harmonic when they are cut by a fifth tangent in harmonic points. Moreover, if four points of a conic are harmonic, the tangents at those points are likewise harmonic (§ 95).

Similar relations exist in other forms of the second order. For example, harmonic rays of a cone of the second order are projected from a fifth ray of the cone by harmonic planes.

Harmonic planes of a pencil of the second order are cut by a fifth plane of the pencil in harmonic rays.

Harmonic rays of a regulus of the second order are cut by any director ray in harmonic points and are projected from any director ray by harmonic planes.

If four rays of a cone are harmonic, the planes tangent to the cone along those rays are likewise harmonic; and if four rays of a regulus are harmonic, the points of contact of planes projecting those rays from any director ray are likewise harmonic.

147. Forms of the First and Second Order in Perspective. The definition of perspective correlation may be extended to forms of the first and second orders as follows.

(*a*) If the points of a conic are projected from any point of the conic by a pencil of rays of

(*b*) Similarly, a pencil of rays of the second order and the range of points in which it is

the first order and each point is correlated to the ray of the pencil passing through it, the conic and the pencil of rays are related perspectively.

The center of the pencil is correlated to the tangent of the conic at that point.

cut by any one of its rays are related perspectively when to each ray of the pencil is correlated the point of the range lying on it.

The ray of the pencil on which the range of points lies is correlated to the point of contact in that ray.

(*c*) A cone of the second order is related perspectively to the pencil of planes of the first order projecting it from any one of its rays, if to each ray of the cone is correlated the plane of the pencil passing through it.

(*d*) A pencil of planes of the second order is perspective to the pencil of rays of the first order in which it is cut by any one of its planes, if to each plane of the pencil of the second order is correlated the ray of the pencil of the first order lying on it.

A regulus of the second order is perspective to the pencil of planes of the first order projecting it from any director ray, or to the range of points of the first order determined by it on any director ray, if each ray of the regulus is correlated to the plane of the pencil passing through it, or to the point of the range lying on it.

Two forms of the second order lying in one another may also be related perspectively, if to each element of the one is correlated the element of the other lying on it or passing through it. For example, a conic lying on a cone or on a regulus of the second order is perspectively related to the form on which it lies if to each point of the conic is correlated the ray of the cone or of the regulus which passes through it. A pencil of rays of the second order is related perspectively to the curve of the second order enveloped by it, if to each ray of the pencil is correlated the point of the

curve lying on it, that is, if to each ray of the pencil is correlated the point of contact in that ray.

In a similar manner, perspective relations may be established between a pencil of planes of the second order and a pencil of rays or a regulus of the second order which it projects; and, in general, a perspective relation may be established between any two forms of the second order in which one form is a section or a projection of the other, or in which one form envelops the other.

148. Forms of the Second Order Projectively Related. As in the case of primitive forms of the first order (§ 50), the following definition may be set up.

Definition. If two forms of the second order are so correlated that to every set of harmonic elements in the one there corresponds a set of harmonic elements in the other, the two forms are said to be correlated projectively.

A form of the first order and a form of the second order are also related projectively if to a set of harmonic elements in the one there corresponds always a set of harmonic elements in the other.

Any two forms perspectively related are likewise projectively related.

As a consequence of the above definition, a sequence of elements in either of two projectively related forms corresponds to a similar sequence of elements in the other (§ 51).

149. Method of Correlating Projectively two Forms of the Second Order. Two forms of the second order may be correlated projectively by establishing a projective relation between two forms of the first order perspective to them. For example, two ranges of points of the second order, that is, the points on two conics or curves of the second order, may be correlated projectively by establishing a projective relation between two pencils of rays of the first order perspective to them.

In establishing such a projective relation, three rays of one pencil and the corresponding rays of the other may be chosen at random (§ 57), and consequently, three points on one conic and the corresponding points on the other may be chosen at random.

If a projective relation is to be established between the ranges of points on the two conics k_1 and k_2, for example, the points A_1, B_1, C_1, on k_1 may be correlated to any three points A_2, B_2, C_2, on k_2 (Fig. 80), and if these are projected from

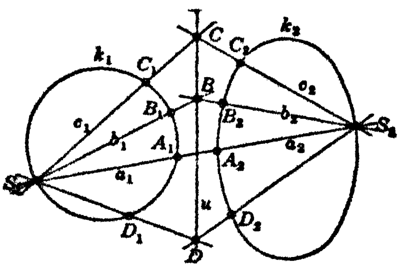


FIG. 80

any points S_1 and S_2 of the respective conics, the problem reduces to the establishment of a projective relation between the pencils of rays S_1 and S_2 in which three pairs of homologous rays are given.

The problem is simplified if the centers S_1 and S_2 are chosen as the points in which the line joining a pair of homologous points, A_1 and A_2, meets the conics a second time, for then the pencils of rays S_1 and S_2 are in perspective relation since their common ray is self-corresponding. If S_1 and S_2 are so chosen, pairs of homologous rays intersect in points of a straight line.

To determine the point of k_2 corresponding, in the required projective relation, to any fourth point D_1 of k_1, if S_1B_1 and S_2B_2, S_1C_1 and S_2C_2, intersect in points, B and C, of a line u, the ray S_1D_1 will determine on the line u a point

D such that the line S_2D will intersect k_2 in the required point D_2.

By this means, to every point of the conic k_1 a corresponding point of k_2 may be determined and the two ranges of points on the conics will be projectively related, since any set of harmonic points of the one will correspond to a set of harmonic points of the other.

The problem of relating other forms of the second order projectively may be reduced to that of two conics, since any form of the second order may be related perspectively to a conic.

In the solution of this problem the following relations are evident.

$$k_1 \overline{\overline{\wedge}} S_1 \overline{\overline{\wedge}} u \overline{\overline{\wedge}} S_2 \overline{\overline{\wedge}} k_2.$$

That is to say, k_1 and k_2, when related projectively, are the first and last of a series of forms in perspective; and, conversely, if two forms are the first and last of a series of perspectivities, they are projectively related.

150. Projective Forms Identical if more than Three Elements are Self-Corresponding. Two conics, k_1 and k_2, in the same plane having a point S in common, may be projected from that point and so be perspective of the same pencil of rays. They are thus projectively related, pairs of homologous points lying on the same ray of the pencil. The point S of k_1 will correspond to the point of k_2 in which the tangent to k_1 at S intersects k_2; and similarly, the point S of k_2 will correspond to the point of k_1 in which the tangent to k_2 at S intersects k_1.

The conics may have three additional points in common which in this correlation will be self-corresponding. If the point S is also self-corresponding, the two conics must have a common tangent at S and, consequently, must coincide, having four points and a tangent at one of them in common.

THEOREM. *If two conics are projectively related and have four self-corresponding points, the conics are identical and all points are self-corresponding.*

THEOREM. *If two conics are projectively related and have four self-corresponding tangents, the conics are identical and all tangents are self-corresponding.*

If a conic is projectively related to a cone or to a regulus of the second order, and more than three points of the conic lie on the rays of the cone or regulus corresponding to them, the conic is perspective to the cone or regulus and to the section of that form made by its plane.

151. Rays of a Pencil of the First Order and Points of a Conic Projectively Related.

THEOREM. *If a pencil of rays of the first order lies in the plane of a conic to which it is projective, but not perspective, at least one ray of the pencil passes through the point of the conic corresponding to it, and at most three rays.*

For if S is the given pencil of rays and S_1 is any pencil of rays of the first order perspective to the given conic k, the center of S_1 lies on k (Fig. 81) and the two pencils S and S_1 generate a conic k_1 different from k, passing through the two centers. The conics k and k_1 intersect at S_1 and necessarily at one other point A, or possibly at three other points A, B, and C, but not more than three. At the points of intersection of k and k_1 other than S_1, a ray of S passes through the point of k corresponding to it. If the conics should have a common tangent at S_1, one of the points, A, B, C, would coincide with S_1 which would then lie on the corresponding ray of S. If the given conic is thought of as

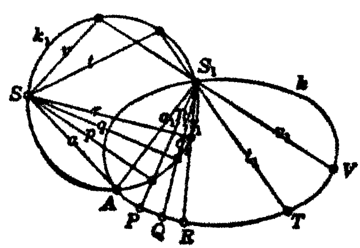


FIG. 81

being generated by two projective pencils of rays, this theorem takes the following form.

THEOREM. *If in a plane there are given three projectively related pencils of rays, no two of which are perspective, there are at most three points of the plane, and at least one, in which homologous rays of the three pencils intersect.*

The reciprocal theorem, of which the proof is analogous to that of the direct theorem, may be stated as follows.

THEOREM. *If a range of points of the first order lies in the plane of a pencil of rays of the second order to which it is projective but not perspective, at least one point of the range will lie on the ray of the pencil corresponding to it, and at most three points.*

If the pencil of rays of the second order is generated by two projective ranges of points of the first order, we have the following theorem.

THEOREM. *If in a plane there are given three projectively related ranges of points, no two of which are perspective, there are at most three straight lines of the plane, and at least one, on which homologous points of the three ranges lie.*

152. Pencils of Rays and Plane Curves of the Third Order. The theorems of § 151 lead to other interesting forms generated by projectively related forms of the first and second orders, of which the following are illustrations.

THEOREM. *If a range of points of the first order and a range of points of the second order lie in the same plane and are projectively related, but not perspective, the lines joining pairs of homologous points in the two ranges form a pencil of rays of which at least one ray and at most three pass through any*

THEOREM. *If a pencil of rays of the first order and a pencil of rays of the second order lie in the same plane and are projectively related, but not perspective, the points of intersection of pairs of homologous rays form a curve or range of points of which at least one point and at most three lie on any line of the plane.*

point of the plane. The pencil of rays so generated is said to be of the third order.

The curve so generated is said to be of the third order.

In the theorem on the left, let S be a pencil of rays perspective to the given range of points of the first order, u, and consequently, projective to the given curve of the second order, k, the center S being any point whatsoever of the plane not lying on u or k. Then, at least one ray of S will pass through the homologous point of k, and at most three (§ 151). In other words, at least one ray joining homologous points in u and k will pass through an arbitrary point S of the plane, and at most three rays joining pairs of homologous points in the two given ranges will pass through that point.

Similarly, on the right, if u is any section of the given pencil of rays of the first order, there are at most three rays of the given pencil of the second order, and at least one, which will pass through the homologous points of u (§ 151). That is, on any line u of the plane there are at most three points of the curve generated by the two forms, and at least one.

153. Space Curves and Pencils of Planes of the Third Order.

THEOREM. *A pencil of planes of the first order and a cone or a regulus of the second order, projectively related, generate, in general, a curve of which at least one point and at most three lie in any plane. Such a curve is called a gauche or twisted curve of the third order.*

THEOREM. *A range of points of the first order and a pencil of rays or a regulus of the second order, projectively related, generate in general a pencil of planes of which at least one plane and at most three pass through any point. Such a pencil of planes is said to be of the third order.*

On the left, an arbitrary plane intersects the given pencil of planes and the cone or regulus of the second order in a pencil of rays and a conic, projectively related, of which at least one and at most three rays of the pencil pass through the homologous points of the conic (§ 151). In such points, a plane of the given pencil intersects its homologous ray of the cone or the regulus. Hence, at least one and at most three points of the curve generated by the two forms lie in this plane.

If the cone or the regulus of the second order is thought of as generated by two projectively related pencils of planes of the first order, the twisted curve of the third order appears as the locus of points in which three homologous planes in the three projective pencils intersect.

On the right, the given pencil of rays or the regulus of the second order is projected from an arbitrary point S by a pencil of planes of the second order (§ 82), and a plane containing the given range of points intersects this pencil of planes in a pencil of rays of the second order, projective to the given range of points, and such that at most three rays of the pencil, and at least one, pass through the homologous points of the given range (§ 151). Hence at most three planes, and at least one, determined by points of the given range and the homologous rays of the given pencil or regulus of the second order pass through an arbitrary point S.

If the pencil of rays or the regulus of the second order is thought of as generated by two projectively related ranges of points of the first order, the pencil of planes of the third order appears as the locus of planes through three homologous points of the three projective ranges.

154. Projectivity on a Conic. Two projectively related ranges of points of the second order may lie on the same conic and form what is called a *projectivity* on the conic. Three elements of the one range and the corresponding

elements of the other may be selected at random and so determine the projectivity.

Suppose A_1, A_2; B_1, B_2; C_1, C_2, are pairs of homologous points in a projectivity on a conic (Fig. 82) in which it is required to determine other pairs of homologous points.

The points of the first range A_1, B_1, C_1, projected from a point A_2 of the second range, and the points A_2, B_2, C_2 of the second range, projected from the corresponding point A_1 of the first range, give pencils of rays $A_2(A_1, B_1, C_1)$ and $A_1(A_2, B_2, C_2)$ which are projectively related, and since the common ray A_1A_2 is self-corresponding, the two pencils are perspective. Pairs of corresponding rays in the two pencils will therefore intersect on the line u determined by the intersections of A_1B_2 and A_2B_1, A_1C_2 and A_2C_1. The point D_2, homologous to any given point D_1 in the projectivity, can then be determined by joining A_2D_1, finding the intersection with u, and joining that point with A_1. The line so drawn will intersect the conic at the required point D_2.

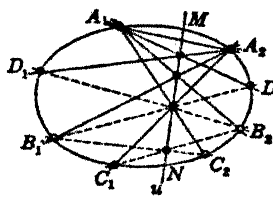


FIG. 82

The figure $A_1B_2C_1A_2B_1C_2$ is a simple hexagon inscribed in the conic, and by Pascal's theorem, the pair of opposite sides B_1C_2 and B_2C_1 will intersect on the line u determined by the intersections of A_1B_2 and A_2B_1; A_1C_2 and A_2C_1, from which it follows that the line u is independent of the pair of homologous points chosen as centers of the projective pencils of rays. All pairs of lines in the projectivity, such as A_1B_2 and A_2B_1, A_1D_2 and A_2D_1, B_1D_2 and B_2D_1, C_1K_2 and C_2K_1, joining homologous points alternately, will consequently intersect on the line u. The line u so determined is called the *axis* of the projectivity.

If the axis intersects the conic in the points M and N,

these points are self-corresponding in the projectivity, and since the axis may intersect the conic in two points, may be tangent to the conic, or may lie wholly outside the conic, the projectivity may have two self-corresponding elements, or one, or none; but always two if the projective ranges of points traverse the conic in opposite senses.

155. Projectivity in a Pencil of Rays of the Second Order. A projectivity in a pencil of rays of the second order may be established in a manner wholly analogous to that of § 154.

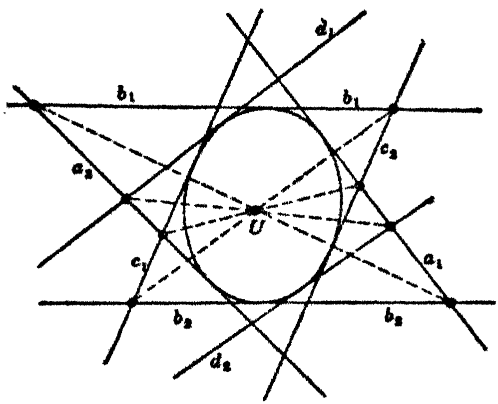


FIG. 83

Three rays of the pencil, a_1, b_1, c_1, and their homologous rays a_2, b_2, c_2 (Fig. 83), may be chosen at random and the projectivity is then fully determined. To any ray d_1, the corresponding ray d_2 may be found and the lines joining pairs of intersections (a_1b_2) and (a_2b_1); (a_1c_2) and (a_2c_1); (b_1c_2) and (b_2c_1), and all lines similarly drawn, pass through one point U called the *center* of the projectivity. The rays of the given pencil, if any, which pass through U, are self-corresponding elements in the projectivity, and there are two such self-corresponding elements, one, or none, according as U lies

outside, on, or inside the conic enveloped by the pencil of rays.

Since the tangents to a conic form a pencil of rays of the second order and the points of a conic are related projectively to the tangents at those points (§ 94), a projectivity on a conic may be transferred directly to a pencil of rays, and in case the projectivities relate to a conic and its tangents, the axis of the projectivity on the conic is the polar of the center of the projectivity in the pencil of rays. In fact, these two projectivities may be considered as a single projectivity connected with the conic, and of this the line u is the axis and its pole U is the center.

156. Projectivity in Other Forms of the First or Second Order. A projectivity on a conic may be transferred to other forms of the first or second order by simple projection and section. For example, the projectivity on the conic k,

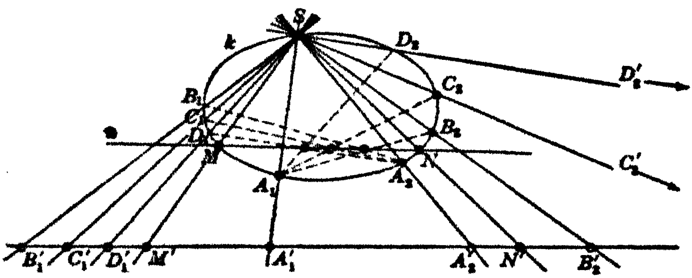


FIG. 84

in which A_1, B_1, C_1, D_1, and A_2, B_2, C_2, D_2, are homologous points and in which M and N are double points (Fig. 84), may be transferred to a straight line by projecting it from any point S of the conic. When so projected on the line, we have such relations as the following:

$$A_1'B_1'C_1'D_1' \barwedge A_2'B_2'C_2'D_2',$$
$$A_1'C_1'M'D_1' \barwedge A_2'C_2'M'D_2',$$
$$A_1'D_1'M'N' \barwedge A_2'D_2'M'N'.$$

The point S is thus the center of two superposed projective pencils of rays in which SA_1 and SA_2, SB_1 and SB_2, $\cdots$ are pairs of corresponding rays, and SM and SN are self-corresponding rays. Similarly, a projectivity on a conic may be transferred to a cone or a regulus or to a pencil of planes of the second order of which the conic or its system of tangents is a section.

157. Projectivities of Special Forms. Special forms of a projectivity on a conic arise according as the points A_1, B_1, C_1 and A_2, B_2, C_2, determining it, are chosen in one way or in another. If A_1 and A_2 coincide, for instance, this point is one of the double elements in the projectivity, and if B_1 and B_2 also coincide, the double elements, the axis, and the center, of the projectivity are determined. If all three points A_1, B_1, C_1 coincide, respectively, with their homologous points A_2, B_2, C_2, the projectivity is identical and all points on the conic are self-corresponding.

158. Projectivity of Doubly Corresponding Elements. In the choice of points A_1, B_1, C_1 and A_2, B_2, C_2, determining a projectivity on a conic, let us suppose that the point A_2 coincides with B_1 and that B_2 coincides with A_1, while C_1 and C_2 are any other pair of homologous points of the two ranges on the conic. The two points of the conic $A_1(B_2)$ and $A_2(B_1)$ in this case are said to correspond to each other *doubly;* that is, they are homologous points whether considered as points of the one range on the conic or as points of the other range.

If the lines A_1C_2 and A_2C_1 (Fig. 85) intersect at P, B_1C_2 and B_2C_1 intersect at Q, the line PQ is the axis of the projectivity determined by the three pairs of homologous points, and if any other point K_1 of the first range on the conic is given, its homologous point K_2 is found by joining K_1A_2 intersecting the axis at R and drawing A_1R cutting the

conic at K_2, homologous to K_1, since K_1A_2 and K_2A_1 intersect on the axis.

If the point C_1 of the first range is D_2, say, of the second, the homologous point of the first range is found by joining B_1D_2 intersecting the axis at P, and drawing B_2P cutting the conic again at D_1, or by joining A_1D_2 intersecting the axis at Q and joining A_2Q cutting the conic at D_1. But in either

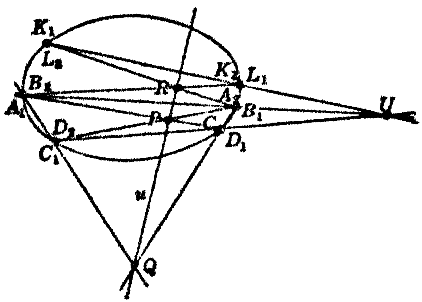


FIG. 85

case, D_1 coincides with C_2; that is, the points of the conic $C_1(D_2)$ and $C_2(D_1)$ are doubly corresponding. In the same way, it may be shown that if any point K_1 of the first range on the conic is L_2 of the second range, then the homologous point K_2 coincides with the homologous point L_1.

From the theory of poles and polars with respect to a conic, it is known that the lines A_1A_2 (or B_2B_1), C_1C_2 (or D_2D_1), K_1K_2 (or L_2L_1) . . . all pass through a point U, the pole of the axis PQ relative to the conic and from this we have the following theorem.

THEOREM. *In a projectivity on a conic, if there is one pair of doubly corresponding points, all pairs of homologous points are doubly corresponding, and the lines joining pairs of such homologous points intersect in one point, the pole of the axis of projectivity. Tangents to the conic at pairs of homologous points intersect on the axis.*

159. Cyclic Projectivity of Period Three. In the choice of points A_1, B_1, C_1, and their homologous points A_2, B_2, C_2, on the conic, suppose A_2 coincides with B_1, B_2 with C_1, and C_2 with A_1 (Fig. 86), so that there is a cyclic relation among them. Then, in determining the axis of projectivity from the pairs of lines A_1B_2 and A_2B_1 and lines similarly drawn (§ 154), A_2B_1 is the tangent at B_1, B_2C_1 is the tangent at C_1, and C_2A_1 is the tangent at A_1, so that the axis is the line on which the sides of the inscribed triangle $A_1B_1C_1$ (or $A_2B_2C_2$) intersect the tangents at the opposite vertices (§ 89).

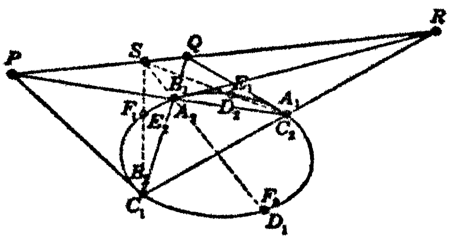


FIG. 86

If D_1 is any other point of the range $A_1, B_1, C_1, \cdots$ on the conic, to determine the homologous point D_2 we join D_1A_2 intersecting the axis at S, and join SA_1 cutting the conic a second time at the required point D_2.

Suppose, now, this point D_2 of the second range on the conic is E_1 of the first range. Since the points C_2 and A_1 coincide, also D_2 and E_1, the line A_1D_2 is the line C_2E_1 cutting the axis at S. Then to determine the homologous point E_2, draw C_1S cutting the conic at the required point E_2.

Again, let E_2 of the second range be F_1 of the first range, and it is required to find the homologous point F_2 of the second range. Since C_1 and B_2 coincide, also E_2 and F_1, the line B_2F_1 is the line C_1E_2 intersecting the axis at the point S. Hence B_1F_2 passes through S and coincides with A_2D_1. Therefore F_2 coincides with D_1.

From this it appears that if A_1, B_1, C_1, and A_2, B_2, C_2, homologous points of a projectivity on a conic, are so related that they are the vertices of an inscribed triangle taken in cyclic order, any other triad of homologous points, D_1, E_1, F_1, and D_2, E_2, F_2, will have the same relation and will be the vertices of a second inscribed triangle. Moreover, in the triangle $D_1E_1F_1$ $(F_2D_2E_2)$ the three sides and the tangents at opposite vertices intersect in points of the axis of projectivity, as do the sides and tangents at opposite vertices of the triangle $A_1B_1C_1$ $(C_2A_2B_2)$. The points of the conic are thus grouped in sets of three which are cyclicly projective, and of such sets there is an unlimited number.

Such a projectivity on a conic may be called a *cyclic projectivity* of period three. The projectivity of § 158 is, then, a cyclic projectivity of period two.

160. Cyclic Projectivity of Period Four. If four points on a conic have a cyclic correspondence in a projectivity on the conic, so that the points A_1, B_1, C_1, D_1 are homologous to the points A_2, B_2, C_2, D_2, respectively, and A_2 coincides with B_1, B_2 with C_1, C_2 with D_1, and finally D_2 with A_1, it is clear that the four points cannot be chosen at random, since three points of one range and the homologous points of the other are sufficient to determine the projectivity.

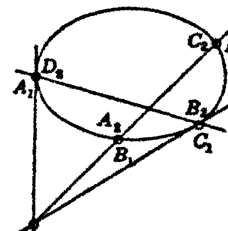


FIG. 87

Suppose the four points and their homologous points in a projectivity on a conic are cyclically related as here defined (Fig. 87). Then A_1D_2 and A_2D_1 intersect on the axis of projectivity, as do also B_1C_2 and B_2C_1. But A_1D_2 is the tangent at A_1, also B_2C_1 is the tangent at C_1, while A_2D_1 and B_1C_2 are the same line, not the axis. Consequently, the tangents at A_1 and C_1, and the line B_1D_1, pass through

the same point of the axis; namely, through the pole of A_1C_1. Hence A_1C_1 and B_1D_1 are conjugate lines, and the points A_1, B_1, C_1, D_1, are harmonic points on the conic (§ 107).

Conversely, the four points of a conic in which a pair of conjugate lines intersect the conic are points of a cyclic projectivity of period four.

For four such points, A_1, B_1, C_1, D_1, are projected from a fifth point P of the conic by four harmonic rays (§ 107), and if A_2, B_2, C_2, D_2 are points homologous to A_1, B_1, C_1, D_1, respectively, in a projectivity on the conic, we have the relation,

$$P(A_1, B_1, C_1, D_1) \barwedge P(A_2, B_2, C_2, D_2).$$

But, by § 34,

$$P(A_1, B_1, C_1, D_1) \barwedge P(B_1, C_1, D_1, A_1).$$

Therefore

$$P(A_2, B_2, C_2, D_2) \barwedge P(B_1, C_1, D_1, A_1).$$

If then A_2 coincides with B_1, B_2 with C_1, and C_2 with D_1, D_2 must coincide with A_1, and the projectivity is cyclic.

EXERCISES

1. If a pencil of rays of the first order is projectively related to a conic, and more than three rays of the pencil pass through the points of the conic corresponding to them, the center of the pencil must lie on the conic and the two forms are in perspective position.

2. Given the two double elements and a pair of homologous points of a projectivity on a conic, determine for any given point of the conic its homologous point.

3. A projectivity between two conics is determined by three tangents of one and their homologous tangents of the other. Construct the tangent homologous to any fourth tangent.

4. Construct the projectivity on a conic of which there are given the axis and one pair of homologous points, the axis being the infinitely distant line of the plane.

5. Construct a cyclic projectivity of period three on a given conic, the axis of projectivity being the infinitely distant line. How many points of such a projectivity may be chosen at random?

CHAPTER XIII

THE THEORY OF INVOLUTION

161. An Involution Defined. If there are given two projective ranges of points u_1 and u_2 on the same straight line, every point of the line is an element of each range, and if the points P_1 of u_1 and Q_2 of u_2 are the same point of the line, ordinarily the corresponding points P_2 of u_2 and Q_1 of U_1 (Fig. 88) are different points of the line. If, however, they should coincide, as in Fig. 89, the two points of the line correspond doubly (§ 158), and if all points of the line are

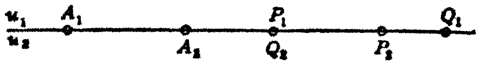


FIG. 88

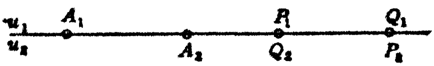


FIG. 89

similarly associated in pairs, the two ranges u_1 and u_2 are ***in involution,*** or they form ***an involution*** on the line.[1] We may therefore state the following definition.

DEFINITION. Two superposed projective forms are said to be in involution when to each element of their common base another element corresponds doubly.

As an illustration of an involution we may consider the pairs of points on any straight line which are polar conjugates relative to a given conic. If $A_1, B_1, C_1, \cdots$ is a range of points on a given line and $A_2, B_2, C_2, \cdots$ are the points on the same line conjugate, respectively, to them, the two

[1] The earliest extended study of the theory of involution is to be credited to Desargues, to whom the term *involution* is due.

ranges of points are projective (§ 110), and if any point A_1 of the first range coincides with B_2 of the second, the conjugate point A_2 of the second range will coincide with B_1 of the first (§ 105), and the same is true for all pairs of conjugate points. The two ranges of conjugate points on the line are therefore in involution.

162. An Involution on a Conic. Two projectively related ranges of points on a straight line may be transferred to a conic by projecting them from any point of the conic (Fig. 90), and the pairs of corresponding points on the conic will have the same relative positions as on the straight line.

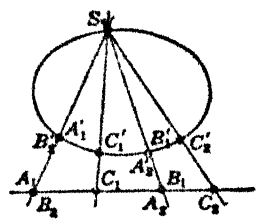


Fig. 90

It has been shown already (§ 158) that if two projective ranges of points on the same conic have one pair of doubly corresponding points, all points of the conic are arranged in pairs which are doubly corresponding, and the form so constituted is a cyclic projectivity on the conic of period two. Such a projectivity on a conic is an involution, since it consists of two superposed

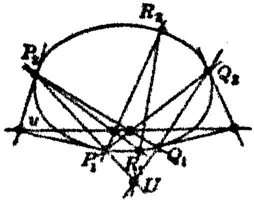


Fig. 91

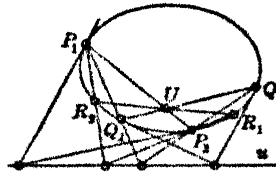


Fig. 92

projective ranges of points in which each point corresponds to another point doubly; and, conversely, any involution on a conic is a cyclic projectivity of period two.

In an involution on a conic, therefore, the pairs of points, P_1, P_2; Q_1, Q_2; R_1, R_2; $\cdots$, each point of a pair correspond-

ing doubly to the other point, are so related that the lines joining them (Fig. 91, 92) pass through one point (§ 158), the socalled ***center of the involution,*** while the lines joining the points of two pairs alternately, P_1Q_2 and P_2Q_1; P_1R_2 and P_2R_1; Q_1R_2 and Q_2R_1; $\cdots$ intersect in points of one straight line, the ***axis of the involution.*** The tangents at the pairs of points, P_1, P_2; Q_1, Q_2; R_1, R_2; $\cdots$ likewise intersect on the axis of the involution.

163. An Involution on a Straight Line. Transferring a cyclic projectivity of period two on a conic back again on a straight line by projecting it from some point of the conic, we have the following result.

THEOREM. *If two superposed projective ranges of points on a straight line have one pair of doubly corresponding points, all pairs are doubly corresponding and the form is an involution.*

In order to determine a projective relation between two ranges of points u_1 and u_2 on a straight line, it is necessary and sufficient to correlate three points of u_1 to three homologous points of u_2 (§ 56) and if two of these points are doubly corresponding, the third points are also doubly corresponding as are all other pairs of corresponding points in the projective ranges, and the form is an involution on the line.

From this it follows that in order to determine an involution on a straight line, it is necessary and sufficient to have given two pairs of associated or conjugate points of the involution.

Suppose there are given two pairs of conjugate points, A_1 and A_2, B_1 and B_2, of an involution on a straight line, and it is required to find the conjugate point in the involution for any given point C_1 of the line. In other words, the problem is to find the point of the straight line which is homologous to a given point C_1 in the projectivity deter-

mined by the three pairs of homologous points A_1, B_1, A_2 and A_2, B_2, A_1, in which A_1 and A_2 correspond doubly.

This may be done most readily by projecting the given points (Fig. 93) from any center S on a conic passing through S, into the points A_1', A_2', B_1', B_2', C_1'. Then the lines $A_1'A_2'$ and $B_1'B_2'$ will intersect in the center U of the involu-

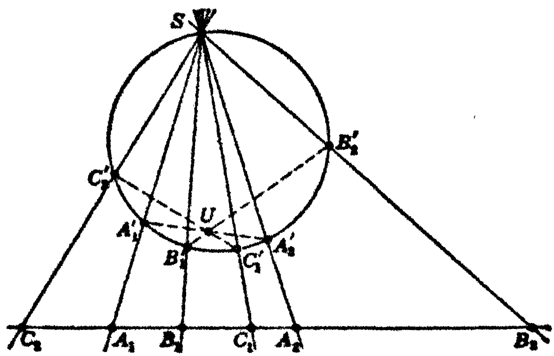


FIG. 93

tion on the conic, and the line UC_1' will intersect the conic a second time at C_2'. This point projected back from S on the original line gives the required point C_2.

164. Involutions in Other Forms of the First or Second Order. An involution may exist in a pencil of rays of the first order as well as in a range of points; in fact, the pencil of rays projecting an involution on a straight line from any center S is such an involution since the rays of the pencil are associated projectively, each ray corresponding to another doubly. The pairs of lines passing through one point which are polar conjugates with respect to a given conic form an involution in a pencil, just as the pairs of polar conjugate points on a straight line form an involution on the line. Also, the pairs of conjugate diameters of an ellipse or a

hyperbola form an involution and so do the pairs of rays through a point which are normal to each other.

Moreover, an involution may exist in a pencil of planes of the first order, as, for example, when the pencil projects an involution in a range of points or in a pencil of rays. So also an involution may exist among the elements of any form of the second order, a pencil of rays, a cone, a pencil of planes, or a regulus, just as it may exist on a conic.

165. Double Elements in an Involution. If the axis of an involution on a conic intersects the conic (Fig. 94), the center of the involution is outside the conic, and two tangents can be drawn from the center to meet the conic in the points in which the axis cuts it. At each of these points,

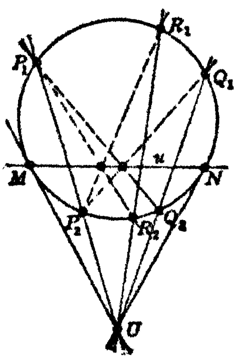


FIG. 94

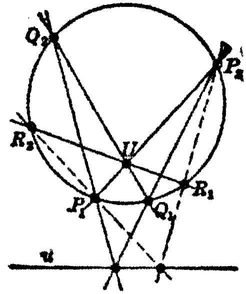


FIG. 95

a pair of conjugate points of the involution coincide and form the so-called *double points* of the involution.

If, however, the axis of the involution does not cut the conic (Fig. 95), the center lies inside, and no tangents from it can be drawn. In this case, the double points do not exist, or, in the language of analytic geometry, they are imaginary.

If the two pairs of points determining an involution on a conic separate each other (Fig. 95), the center of the involution is inside the conic, and the double points are imaginary. Such an involution is called ***elliptic***. In an elliptic involution any two pairs of conjugate points separate each other.

If the two pairs of points determining the involution do not separate each other (Fig. 94), the center lies outside the conic and the double points may be found; in other words, the double points are real. Such an involution is called ***hyperbolic*** and in it no pair of conjugate points is separated by any other pair.

Any line through the center of an involution on a conic is a polar conjugate of the axis of the involution. Hence, in a hyperbolic involution if a line is drawn through the center intersecting the axis inside the curve, this line and the axis cut the conic in four harmonic points (§ 107). From this we have the following theorem.

THEOREM. *All pairs of conjugate points of an involution on a conic are harmonically separated by the double points when these are real.*

By projecting an involution on a conic from any point of the conic upon a straight line we obtain the following relations.

(*a*) An involution on a straight line has real double points if the two pairs of conjugate points determining it do not separate each other. In this case, the involution is hyperbolic and no pair of conjugate points is separated by another pair. All pairs of conjugate points are harmonically separated by the double points.

(*b*) An involution on a straight line has no (real) double points, or we may say the double points are imaginary, if the two pairs of conjugate points determining it separate each other. In this case, the involution is elliptic and every pair of conjugate points is separated by every other pair.

166. The Center of an Involution on a Straight Line. To every point of an involution on a straight line there is a conjugate point. The point of the line conjugate to the infinitely distant point is called the *center* of the involution.[1] When the double points of the involution are real, the center is midway between them.

167. Six Points of a Line in Involution. Three pairs of points on a straight line, A_1,A_2; B_1,B_2; C_1,C_2, are said to be in involution when the third pair, C_1, C_2, belongs to the involution determined by the other two pairs; that is to say, when the sixth point C_2 is the conjugate of C_1 in the involution determined by A_1, A_2, and B_1, B_2.

If the three pairs of points are in involution, any four of them are projective to their four conjugates. For example, among other relations, we may write

$$A_1B_1B_2C_1 \barwedge A_2B_2B_1C_2,$$

and also

$$A_2B_1C_1C_2 \barwedge A_1B_2C_2C_1.$$

If M and N are the double points of this involution, each of them is its own conjugate, and we have the relations

$$A_1MB_1C_1 \barwedge A_2MB_2C_2,$$
$$B_1MC_2N \barwedge B_2MC_1N.$$

Conversely, if three pairs of points on a straight line, A_1, A_2; B_1, B_2; C_1, C_2, are so related that four of the points involving all three pairs are projective to their four conjugates in order, the points of one pair in the projective forms being thus doubly corresponding, the six points form three pairs of an involution.

For example, if $A_1B_2C_1C_2$ are projective to $A_2B_1C_2C_1$, a relation in which A_1 is homologous to A_2, B_2 to B_1, C_1 to C_2,

[1] The center of an involution on a straight line must not be confused with the center of an involution on a conic or with the center of a projectivity of period two on a conic.

and C_2 to C_1; that is, a projective relation in which the points C_1 and C_2 correspond doubly, then A_1, A_2; B_1, B_2; C_1, C_2, form an involution (§ 163).

168. Metric Properties. If A_1, A_2; $B_1 B_2$; C_1, C_2, are pairs of points of an involution on a straight line, we have, among other relations,

$$A_1 B_2 C_1 C_2 \barwedge A_2 B_1 C_2 C_1.$$

Since pairs of conjugate points in this projective relation correspond doubly, any two conjugate points may be interchanged, and the projective relation will still hold true. Moreover, any cross-ratio of the first four points equals the similar cross-ratio of the second four (§ 59). That is to say,

$$\frac{A_1B_2 \cdot C_1C_2}{A_1C_1 \cdot B_2C_2} = \frac{A_2B_1 \cdot C_2C_1}{A_2C_2 \cdot B_1C_1}.$$

Dividing by C_1C_2 and clearing of fractions we have the relation

$$A_1B_2 \cdot A_2C_2 \cdot B_1C_1 = -A_2B_1 \cdot A_1C_1 \cdot B_2C_2.$$

Interchanging C_1 and C_2 and reversing certain segments gives the formula

$$A_1B_2 \cdot B_1C_2 \cdot C_1A_2 = -A_2B_1 \cdot B_2C_1 \cdot C_2A_1.$$

This formula expresses a relation which holds true for any three pairs of points in involution.

The preceding relation among the segments determined on a straight line by three pairs of points in involution may be rewritten and reduced to the form

$$\frac{A_1B_2 \cdot C_1A_2}{A_2B_1 \cdot B_2C_1} = \frac{A_1C_2}{B_1C_2} = \frac{A_1B_1 + B_1C_2}{B_1C_2} = 1 + \frac{A_1B_1}{B_1C_2}.$$

If, now, the point C_2 becomes infinitely distant, its conjugate point C_1 is the center of the involution on the straight line, and in this position we shall designate it by the letter O. When C_2 is at infinity on the line, the fraction A_1B_1/B_1C_2 becomes indefinitely small and the above relation, replacing

C_1 by O, may be written in the form

$$A_1B_2 \cdot OA_2 = -A_2B_1 \cdot OB_2,$$

or

$$(A_1O + OB_2)OA_2 = -(A_2O + OB_1)OB_2.$$

This relation simplifies into the formula

$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2,$$

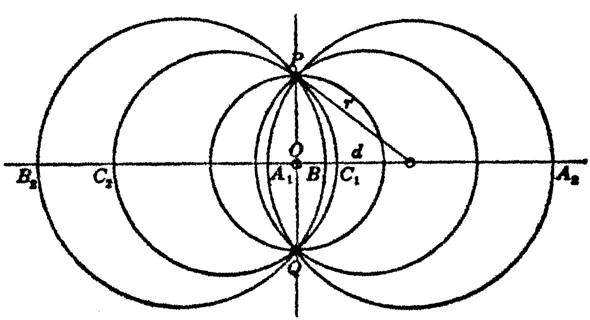
where A_1, A_2; B_1, B_2, are any pairs of conjugate points in the involution and O is the center of the involution. This relation is usually taken as the metric definition of an involution.

If M and N are the double points of the involution, any pair of conjugate points in this formula may be replaced by either M or N, so that we have the relations

$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = OP_1 \cdot OP_2 = OM^2 = ON^2.$$

If the involution is elliptic, that is, if the pairs of conjugate points separate each other, OA_1 and OA_2, OB_1 and OB_2, and all such pairs of line-segments have opposite signs and their product is negative, so that OM^2 has no real meaning, and the point M does not exist. This result is in accord with the statement of § 165.

169. Co-Axial Circles. Suppose that on the line-segments A_1A_2, B_1B_2, C_1C_2, $\cdots$ in any involution on a straight line, of



which O is the center, circles are described with these segments as diameters. If the involution is elliptic, the line-segments overlap (Fig. 96), any two of the circles intersect, and the point O lies inside all of the circles. If, however, the involution is hyperbolic, the circles do not intersect (Fig. 97) and O is outside all of the circles.

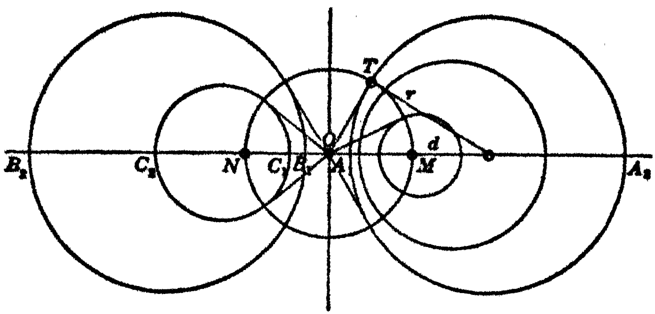


FIG. 97

If d is the distance from O to the center of the circle described on the segment A_1A_2, and r is the radius of that circle, then, for either Fig. 96 or 97, we have

$$OA_1 = d - r,$$

and

$$OA_2 = d + r.$$

Therefore

$$OA_1 \cdot OA_2 = d^2 - r^2.$$

If d' and r' are the corresponding distances for the circle whose diameter is B_1B_2, we have

$$OB_1 \cdot OB_2 = d'^2 - r'^2,$$

and similar relations hold for all the circles.

In an elliptic involution (Fig. 96), $d^2 - r^2$ is negative since OA_1 and OA_2 have opposite signs, but $r^2 - d^2$ being positive equals the square of the distance from O to either

of the points in which the circle on A_1A_2 is cut by a line through O perpendicular to the line of centers. Since

$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = OC_1 \cdot OC_2 = \cdots$$

for the several circles, this distance is the same for all circles, and, consequently all the circles described on the segments A_1A_2, B_1B_2, C_1C_2, $\cdots$ in an elliptic involution, cut the line through O perpendicular to the line of centers, in the same two points.

If the given involution is hyperbolic (Fig. 97), $OA_1 \cdot OA_2$, and similar products for the other circles, is positive, that is, $d^2 - r^2$ is positive, d is greater than r, and O lies outside the circle, as has already been pointed out. In this case, if a tangent is drawn from O to any one of the circles meeting it at T, $d^2 - r^2 = OT^2$. Hence, the tangents drawn from O to the several circles are all equal and the circle with center O and radius equal to $d^2 - r^2$ passes through the points of contact of all such tangents.

Since $OA_1 \cdot OA_2 = OM^2 = ON^2$ in the hyperbolic involution, $d^2 - r^2 = OM^2 = ON^2$ and, consequently, the circle whose center is O and radius OM (or ON) will pass through all the points of contact T of the several circles on A_1A_2, B_1B_2, $\cdots$ as diameters and will cut all the circles of the system orthogonally.

The two systems of circles here described are said to be *co-axial* and the line through O, perpendicular to the line of centers on which the involution lies, is called the *radical axis* of any two of the circles. In case the involution is elliptic, the circles all pass through the same two points on the radical axis. If the involution is hyperbolic, the circles form nests on either side of the radical axis, each nest with a limiting point, or circle of zero radius, these being the double elements of the involution. The circle with center O which passes through the limiting points cuts the circles of the hyperbolic system orthogonally.

If P and Q are the points on the radical axis in which the circles intersect whose diameters are the segments A_1A_2, B_1B_2, C_1C_2, $\cdots$ of an elliptic involution, the involution is projected from either P or Q by a pencil of rays in involution in which each ray is normal to its conjugate ray. From this we have the following theorem.

THEOREM. *In the plane of an elliptic involution on a straight line, there are two points from which the pairs of conjugate points of the involution are projected by rays at right angles. These points lie on the line through the center of the involution perpendicular to the line on which the involution itself lies.*

170. One Pair of Conjugate Rays at Right Angles in any Involution Pencil. If a_1, a_2, and b_1, b_2, are two pairs of conjugate rays in a given involution in a pencil whose center is S (Fig. 98), a circle may be described through S cutting the pairs of rays at points A_1, A_2, and B_1, B_2, respectively, and by this means the given involution is transferred to the circle. The lines A_1A_2 and B_1B_2 intersect in U, the center of the involution on the circle, and any line through U cutting the circle will determine two points, C_1 and C_2, on the circle such that SC_1 and SC_2 are conjugate rays of the given involution. The line through U which passes also through the center of the circle will determine points K_1 and K_2 on the circle such that SK_1 and SK_2 are conjugate rays of the involution and at the same time are at right angles to each other since K_1K_2 is a diameter.

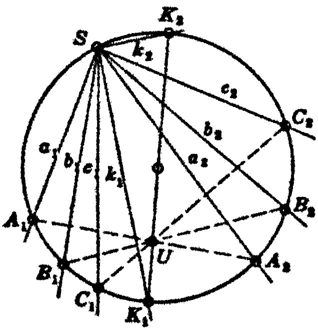


FIG. 98

The line K_1K_2 through the center of the involution and through the center of the circle can always be drawn, so that there is always one pair of conjugate rays of the involution at right angles, but if more than one diameter of the circle passes through the center of the involution, this center must coincide with the center of the circle and all pairs of conjugate rays through S are at right angles. From this property we have the following theorem.

THEOREM. *In an involution in a pencil of rays there is always one pair of conjugate rays at right angles and if there is more than one pair of conjugate rays at right angles, then all pairs are at right angles.*

171. An Involution on a Tangent to a Conic.

THEOREM. *If an involution is chosen on a given tangent to a conic and the other tangents to the conic are drawn from the pairs of conjugate points of this involution, these pairs of tangents will intersect in points of a straight line and the lines joining their points of contact will pass through one point.*

Let us suppose that on a given tangent a of a conic (Fig. 99) there is an involution A_1, A_2; B_1, B_2; C_1, C_2, and that from the points of this involution the second tangents to the conic are drawn, their points of contact being the points A_1', A_2'; B_1', B_2'; C_1', C_2', respectively. The theorem states that the tangents at the three pairs of points A_1', A_2'; B_1', B_2'; C_1', C_2', intersect in points of one straight line and that the three lines $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$, pass through one point.

Since the points of contact of tangents to a conic are projective to the points of intersection of these tangents with a fixed tangent (§ 94), the points A_1', A_2'; B_1', B_2'; C_1', C_2', form an involution on the conic and the lines joining pairs of homologous points in this involution pass through one point; namely, the center of the involution on the conic. At the same time, the tangents at these pairs of points

intersect on the axis of the involution on the conic (§ 162), the polar of the center.

If the involution on the tangent a has double points, the corresponding involution on the conic will also have double points and the axis on which pairs of conjugate tangents

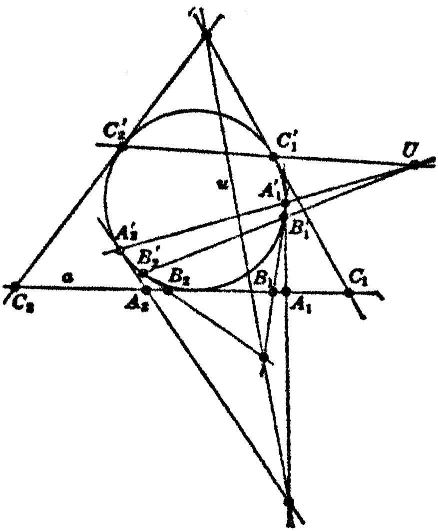


Fig. 99

intersect will cut the conic at these points and the center of the involution will lie outside the conic; otherwise, the axis will not cut the conic and the center will lie inside.

172. Orthogonal Tangents to a Parabola. As an application of the theorem of § 171, the following may be stated.

Theorem. *The pairs of tangents to a parabola which are at right angles to each other intersect on a fixed straight line and the chords of contact of these tangents pass through one point.*

Lines through any point of the plane parallel to these tangents form an involution (§ 164) which determines an involution on the infinitely distant line of the plane. Hence,

the pairs of tangents at right angles are drawn from pairs of homologous points of an involution on the infinitely distant tangent of the parabola. These pairs of conjugate tangents will therefore intersect in points of one straight line and their chords of contact will pass through one point, the pole of that line.

The straight line on which the pairs of orthogonal tangents to a parabola intersect is the *directrix* of the parabola for which a more general definition is given in § 201.

173. The Sides of a Complete Quadrangle are cut by a Straight Line in an Involution. In § 60 it was shown that four points on a straight line are projective to any permutation of those points in which one pair and also the other pair are interchanged. For example, if A, B, C, D, are any four points on a straight line, we have the relations

$$ABCD \barwedge BADC \barwedge CDAB \barwedge DCBA.$$

By making use of this property, the following theorems may be proved.

THEOREM. *The three pairs of opposite sides of a complete quadrangle are cut by any straight line not passing through a vertex in three pairs of points in involution.*

THEOREM. *The three pairs of opposite vertices of a complete quadrilateral are projected from any point not lying on a side by three pairs of rays in involution.*

The theorem on the left is due to Desargues[1] and the theorem on the right is its direct reciprocal. In the theorem on the left, if the line u (Fig. 100) intersects the pairs of opposite sides of the quadrangle $PQRS$ in the points A_1, A_2; B_1, B_2; C_1, C_2, the point O being the intersection of PR and QS, and if we project first from Q and then from S, we have the

[1] Works of Desargues, edited by Poudra, Paris, 1864, Vol. I, p. 171.

relations

$$C_2A_1C_1B_2 \barwedge C_2POR \barwedge C_2B_1C_1A_2.$$

By permutation, we have

$$C_2A_1C_1B_2 \barwedge C_1B_2C_2A_1.$$

Therefore

$$C_1B_2C_2A_1 \barwedge C_2B_1C_1A_2.$$

In this projective relation, C_1 and C_2 correspond doubly, while A_1 corresponds to A_2 and B_1 to B_2.

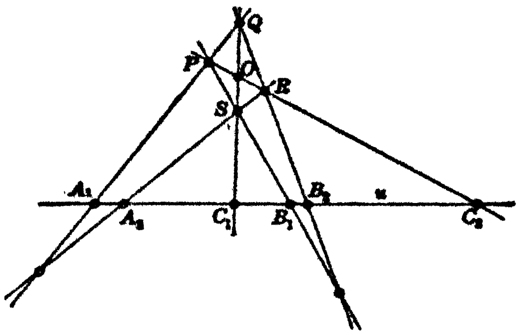


FIG. 100

Therefore A_1, A_2; B_1, B_2; C_1, C_2, form an involution (§ 163). The theorem on the right may be proved analogously.

If the line u passes through the intersection of a pair of opposite sides of the quadrangle, one pair of conjugate points in the involution coincide and become a double point of the involution; if the line passes through the intersections of two pairs of opposite sides, these are the two double points and the remaining pair of conjugate points in the involution are harmonically separated by them.

174. Construction for Points in Involution. The left-hand theorem of § 173 may be used to determine the conjugate to any fifth point in an involution on a straight line

when two pairs of conjugate points are given. By such use, the required point can be found without projecting the given points on a conic as in § 163.

If A_1, A_2; B_1, B_2 are two pairs of points in an involution on a line and it is required to find C_2 the conjugate to any given fifth point C_1, we may proceed in the following manner. Through A_1 and A_2 draw a pair of opposite sides PQ and RS of a quadrangle (Fig. 100); through C_1 draw a diagonal QS; through B_1 and B_2 draw a second pair of opposite sides PS and QR; and the second diagonal PR will determine on the given line the required point C_2.

This construction, however, does not lend itself readily to the location of the double points of the involution. To find those points resort may be had to § 165 or to the construction of § 40, either of which yields a unique pair of points, if such points exist, which harmonically separate both of the given pairs and which consequently are the double points of the involution determined by those pairs.

175. The Mid-points of the Diagonals of a Complete Quadrilateral are Collinear. The theorem on the right of § 173 may be utilized for the establishment of the following interesting general theorem.

THEOREM. *In a given complete quadrilateral, if the sides of the diagonal triangle are intersected by a straight line in the points P, Q, R, and the harmonic conjugates of these points, P_1, Q_1, R_1, are found relative to the two vertices of the quadrilateral lying on the same side of the triangle, the points P_1, Q_1, R_1 are collinear.*

Suppose A_1, A_2; B_1, B_2; C_1, C_2 (Fig. 101) are the three pairs of opposite vertices of the given quadrilateral and that the line u intersects the side A_1A_2 of the diagonal triangle in the point P, while P_1 on the same side is such that A_1, P, A_2, P_1 are harmonic. Similarly, suppose B_1, Q, B_2, Q_1 and C_1, R, C_2, R_1 are harmonic, the points P, Q, R,

lying on the line u. The theorem states that P_1, Q_1, R_1 are collinear.

Let the line P_1Q_1, or u_1, intersect the line u at a point O and from O project the pairs of opposite vertices A_1, A_2; B_1, B_2; C_1, C_2 of the quadrilateral by the rays a_1, a_2; b_1, b_2; c_1, c_2, respectively. These pairs of rays are in involution (§ 173) and since the lines u and u_1 separate the rays a_1, a_2, also the rays b_1, b_2, harmonically, they are the double rays of the involution and separate all pairs harmonically (§ 165). Hence they separate c_1 and c_2 harmonically. Since the rays c_1, u, c_2 pass through the points C_1, R, C_2, respectively, the ray u_1 must pass through R_1. In other words, P_1, Q_1, R_1 are collinear.

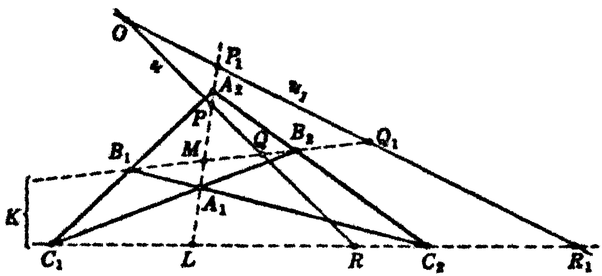


FIG. 101

From this general property the following theorem may be deduced as a special case.

THEOREM. *In any complete quadrilateral, the mid-points of the three diagonals lie on one straight line.*

If the chosen line u in the general case is the infinitely distant line of the plane, the points P_1, Q_1, R_1 are the mid-points of the diagonals of the given quadrilateral.

§ 176. Desargues' Theorem on an Inscribed Quadrangle. Among the many advances in pure geometry attributable to Desargues, the following theorem and the theorem on perspective triangles (§ 24) commonly bear his name.

THEOREM. *If a quadrangle is inscribed in a conic, any straight line not passing through a vertex intersects the conic and the pairs of opposite sides of the quadrangle in an involution.*

Let $PQRS$ be a quadrangle inscribed in a given conic (Fig. 102) and let a straight line u intersect two pairs of opposite sides of the quadrangle in points A_1, A_2 and B_1, B_2,

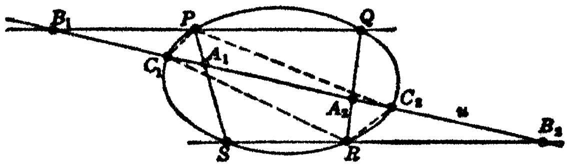


FIG. 102

and intersect the conic in points C_1 and C_2. Then, A_1, A_2; B_1, B_2; C_1, C_2, is an involution. For, projecting from the vertices P and R, we have the relation

$$P(C_1SC_2Q) \barwedge R(C_1SC_2Q).$$

Hence, of the points on the line u,

$$C_1A_1C_2B_1 \barwedge C_1B_2C_2A_2.$$

By permutation, we have

$$C_1A_1C_2B_1 \barwedge C_2B_1C_1A_1.$$

Therefore

$$C_1B_2C_2A_2 \barwedge C_2B_1C_1A_1.$$

From this it follows that A_1, A_2; B_1, B_2; C_1, C_2, is an involution (§ 163). The third pair of opposite sides of the quadrangle, PR and QS, determine another pair of conjugate points in this involution, as does also any other conic circumscribing the quadrangle.

177. Conics through Four Points. The involution in §176 will be hyperbolic or elliptic according as the line u is drawn in one position or another relative to the vertices of the quadrangle. If the involution is hyperbolic, a pair of con-

jugate points coincide at each of the double points and a conic passing through the vertices of the quadrangle is tangent to u at each of these points.

If the line u is so drawn that the involution on it, determined by the quadrangle or by any two conics through the four vertices, is elliptic, no conics of the system are tangent to the line, since in the involution there are no double points. If the line passes through one of the vertices of the quadrangle, that is, through one of the four points common to the conics, there is one conic of the system tangent to the line at that point and none elsewhere, since the line intersects all other conics of the system at that point. Hence the following theorem may be stated.

THEOREM. *Of the system of conics through four points, there are, in general and at most, two which are tangent to any straight line not passing through one of the points.*

Of the conics through four points two are parabolas since, in general, two of the conics are tangent to the infinitely distant line. If, however, the four points are so situated relative to each other that one of them lies within the triangle formed by the other three, the involution on the infinitely distant line determined by the pairs of opposite sides of the quadrangle of which these four points are vertices is elliptic and no parabolas pass through the four points. In fact, all conics through four points so related are hyperbolas since, no matter how the fifth point may be chosen to determine a conic, two of them may be selected as centers which will make the generating pencils oppositely projective.

178. A Quadrilateral Circumscribed to a Conic.

THEOREM. *If a quadrilateral is circumscribed to a given conic and tangents to the conic are drawn from any point not on a side of the quadrilateral, these tangents and the rays through the point projecting pairs of opposite vertices of the quadrilateral form an involution.*

Let p, q, r, t (Fig. 103) be the sides of the circumscribed quadrilateral and let S be the point from which tangents c_1 and c_2 are drawn to the given conic. If the rays a_1 and a_2, b_1 and b_2, are drawn from S projecting pairs of opposite vertices of the quadrilateral, the rays a_1, a_2; b_1, b_2; c_1, c_2, form an involution. For the tangents p and r intersect

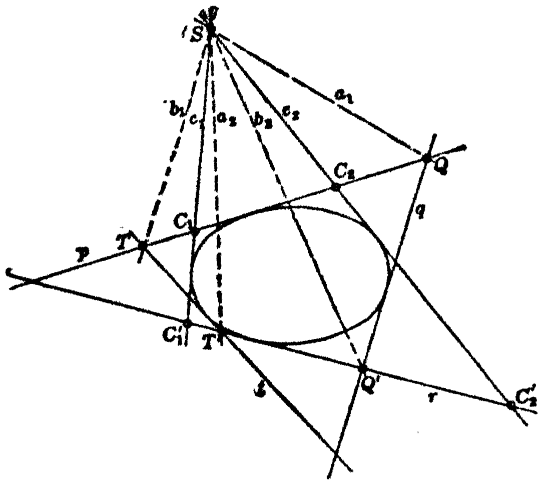


FIG. 103

the other tangents of the conic q, t, c_1, c_2, in ranges of points which are projectively related (§ 66). That is to say, QTC_1C_2 are projective to $Q'T'C_1'C_2'$. Projecting these points from S gives

$$S(QTC_1C_2) \barwedge S(Q'T'C_1'C_2')$$

or

$$a_1b_1c_1c_2 \barwedge b_2a_2c_1c_2.$$

By permutation, we have

$$b_2a_2c_1c_2 \barwedge a_2b_2c_2c_1.$$

Therefore

$$a_1b_1c_1c_2 \barwedge a_2b_2c_2c_1.$$

In this projective relation, the rays c_1 and c_2 correspond doubly. Hence a_1, a_2; b_1, b_2; c_1, c_2, form an involution (§ 163). The rays projecting the third pair of opposite vertices of the quadrilateral from S are conjugate rays in this involution, as are also the pairs of tangents drawn from S to any other conic inscribed in the same quadrilateral.

From this it follows that if a system of conics is inscribed in a given quadrilateral, tangents to the several conics drawn from any point S, not on a side of the quadrilateral, form an involution to which belong also the pairs of rays projecting opposite vertices of the quadrilateral.

If the point S is so chosen that this involution is hyperbolic, two double elements appear in the involution, each of which is a pair of coincident tangents from S to one of the inscribed conics. The point S therefore lies on two conics of the system and to each of these a double ray of the involution is tangent. If the point S is so chosen that the involution is elliptic, no conic of the system passes through S.

THEOREM. *Of the conics inscribed in a given quadrilateral, in general and at most, two pass through a given point.*

179. Pairs of Opposite Sides of a Complete Quadrangle Orthogonal. The infinitely distant line of the plane cuts the sides of any given quadrangle in an involution (§ 173); hence, if through any point of the plane lines are drawn parallel to the three pairs of opposite sides of a complete quadrangle, they form an involution. If two pairs of the lines so drawn are at right angles, the third pair must also be at right angles (§ 170), from which we have the following theorem.

THEOREM. *If two pairs of opposite sides of a complete quadrangle are at right angles, the third pair is also at right angles.*

From this theorem it follows immediately that the perpendiculars from the vertices of a triangle on the opposite

sides meet in a point. For if lines from two vertices of the triangle are drawn perpendicular to the opposite sides, the line through their intersection and the third vertex may be taken with the third side of the triangle to form the third pair of opposite sides of a complete quadrangle of which two pairs are at right angles. The point of intersection of the three perpendiculars is the *orthocenter* of the triangle.

180. Conics through the Vertices and the Orthocenter of a Triangle. If a point D is the orthocenter of a triangle ABC, then A is the orthocenter of the triangle BCD, as are also B and C the orthocenters of the triangles formed by the remaining three vertices.

THEOREM. *All conics through the vertices of a triangle and its orthocenter are rectangular hyperbolas.*

Since one of the points necessarily lies within the triangle formed by the other three, the conics through the four points are hyperbolas (§ 177). The infinitely distant line of the plane cuts the system of conics through the four points and the sides of the inscribed quadrangle in an involution (§ 176) which if projected from any point of the plane will give an involution pencil of rays. In this involution all pairs of conjugate rays are at right angles (§ 170). And since any pair of conjugate rays of the involution which project the infinitely distant points of a conic of this system determine the direction of the asymptotes of that conic, the asymptotes of each of these hyperbolas are at right angles and the hyperbolas are rectangular.

181. Through Four Points of a Plane there is One Rectangular Hyperbola.

THEOREM. *Through the vertices of any quadrangle in a plane, at least one, and in general only one, rectangular hyperbola can be drawn.*

In the involution determined by lines through a point of the plane parallel to the pairs of opposite sides of the given

quadrangle, the pairs of conjugate rays give the directions of the asymptotes of the hyperbolas in the system of conics through the vertices of the quadrangle. Among these there is always one pair of conjugate rays at right angles (§ 170) and, consequently, there is one rectangular hyperbola in the system. If there is more than one pair of conjugate rays at right angles, all pairs are at right angles and all conics through the vertices of the quadrangle are rectangular hyperbolas. In this case the pairs of opposite sides of the given quadrangle are at right angles, and each vertex is the orthocenter of the triangle formed by the other three vertices of the quadrangle.

182. Properties of the Rectangular Hyperbola through Four Given Points.

THEOREM. *The rectangular hyperbola through the vertices of an arbitrary quadrangle passes also through the orthocenters of the four triangles determined by these vertices.*

For the rectangular hyperbola through the vertices A, B, C, D, of a given quadrangle will have two points P_1 and P_2 on the infinitely distant line and these points determine the directions of the asymptotes of this hyperbola. If K is the orthocenter of the triangle ABC and a second conic is drawn through A, B, C, K, and P_1, it is a rectangular hyperbola (§ 180) of which P_1 determines the direction of one asymptote, and consequently P_2 determines the direction of the other asymptote. This second conic, therefore, coincides with the first since it has in common with it the points A, B, C, P_1 and P_2. Hence the rectangular hyperbola through the four points A, B, C, D, passes also through K, the orthocenter of the triangle ABC. It likewise passes through the orthocenters of the triangles ABD, ACD, and BCD. Incidentally, the following theorem may be stated.

THEOREM. *If a triangle is inscribed in a rectangular hyperbola, its orthocenter lies on the hyperbola.*

183. Orthocenters of Triangles Formed by Four Lines are Collinear. The three sides of any triangle and the infinitely distant line of the plane form a quadrilateral of which the three pairs of opposite vertices are projected from the orthocenter of the triangle by pairs of rays at right angles, determining an involution (§ 164). If the sides of the triangle are tangent to a parabola, the pair of tangents to this parabola which can be drawn from the orthocenter belong to this involution (§ 178) and, consequently, they are at right angles. The orthocenter, therefore, of any triangle circumscribed to a parabola lies on the directrix of the parabola (§ 172).

Since one and only one parabola may be inscribed in a given quadrilateral, the following theorem may be stated.

THEOREM. *The orthocenters of the four triangles formed by the sides of a quadrilateral lie on one straight line; namely, on the directrix of the parabola inscribed in the quadrilateral.*

184. Tangents to a Conic from Pairs of Points of an Involution.

THEOREM. *If in the plane of a conic there is given an involution on any straight line not cutting the conic and tangents are drawn to the conic from the pairs of conjugate points of the involution, these pairs of tangents intersect in general on a second conic* (compare § 171).

If A_1 and A_2 are a pair of conjugate points in the given involution on a fixed line u not cutting a given conic k, from each of them there can be drawn two tangents to the conic (Fig. 104) forming a circumscribed quadrilateral of which the pairs of opposite vertices, other than A_1 and A_2, are Q_1 and Q_2, R_1 and R_2. From another pair of conjugate points of the given involution, B_1 and B_2, a pair of tangents may be drawn to the conic intersecting at S. The rays SA_1 and SA_2, SR_1 and SR_2, SQ_1 and SQ_2, SB_1 and SB_2, are in involution (§ 178), and consequently SR_1 and SR_2 will inter-

sect the given line u in points C_1 and C_2 of the involution determined by A_1, A_2, and B_1, B_2, as will also SQ_1 and SQ_2. Conversely, if R_1C_1 and R_2C_2 intersect in S, C_1 and C_2 being any pair of conjugate points of the given involution, the tangents to the conic drawn from S will intersect u in a pair of conjugate points of the given involution.

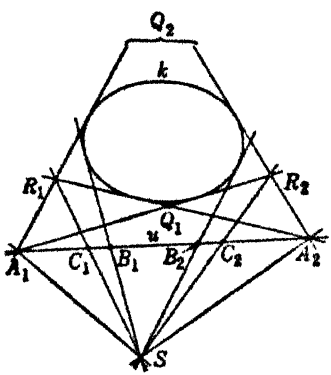


FIG. 104

If now R_1 and R_2 are chosen as centers of pencils of rays and those rays in the two pencils are correlated which pass through pairs of conjugate points of the given involution on u, the pencils will be projectively related and a pair of homologous rays R_1C_1 and R_2C_2 will intersect in S which is also the point of intersection of tangents to the conic from a pair of conjugate points B_1 and B_2 in the involution. Hence it is the locus of S which is required.

The pencils of rays R_1 and R_2 generate a conic k_1 which is the locus of S and which passes through R_1 and R_2, also through Q_1 and Q_2, the intersections, respectively, of the corresponding rays R_1A_2 and R_2A_1 and the corresponding rays R_1A_1 and R_2A_2. This conic k_1 also passes through the self-conjugate points of the given involution, if such there are, and since these self-conjugate points harmonically separate all pairs of conjugate points of the involution, the theorem of this article may be stated in the following form.

THEOREM. *The pairs of tangents to a conic which are harmonically separated by two fixed points intersect, in general, on a second conic.*

The reciprocal theorem, the proof of which is analogous to that here given, may be stated as follows.

THEOREM. *If in the plane of a conic there is given a pencil of rays in involution whose center lies inside the conic, the lines joining the points in which pairs of conjugate rays intersect the conic, envelop a second conic.*

If the center of the involution lies on the given conic, the lines joining points in which conjugate rays intersect the conic all pass through one point. This case is reciprocal to the theorem of § 171.

185. Locus of the Intersection of Orthogonal Tangents to a Conic. In the diagram of § 184 (Fig. 104), if the angles A_1SA_2 and R_1SR_2 (C_1SC_2) are right angles, as they may be (§ 169), the involution determined at S is rectangular and the tangents drawn from S are at right angles. In this case, the conic described by S is the locus of the intersection of tangents to the given conic, at right angles to each other, and is a circle of which R_1R_2 is a diameter; likewise, Q_1Q_2 is a diameter. The figure $R_1Q_1R_2Q_2$ is therefore a rectangle circumscribing the given conic k, and the locus of S is a circle through the vertices of this rectangle.

If the given conic is a parabola, the discussion of this paragraph does not apply, since a parabola has no actual tangents parallel to each other and hence no circumscribing rectangle; neither does it apply if the given conic is a hyperbola in which the semi-axis b is greater than the semi-axis a; that is, if the angle between the asymptotes in which the hyperbola lies is greater than a right angle, for in that case the hyperbola has no tangents which are at right angles to each other. The following theorem may therefore be stated.

THEOREM. *If two tangents to an ellipse or a hyperbola are at right angles, the locus of their intersection is a circle.*

This circle is known as the ***director circle*** of the conic.

186. The Director Circle of an Ellipse or a Hyperbola. The theorem of § 185 may be given a direct proof as follows.

Let $ABCD$ be a rectangle circumscribed to a given ellipse or hyperbola (Fig. 105) and let S be any point on the circle passing through the vertices of this rectangle. The pairs of lines drawn from S to opposite vertices of the rectangle and the tangents from S to the conic form an involution (§ 178) in which the two pairs of conjugate rays to opposite vertices are at right angles. Consequently, the two tangents from S to the conic; that is, tangents from any point of the circle to the conic, are at right angles (§ 170), while tangents to this conic from any point not on the circle cannot be at right angles. This locus of the intersection of orthogonal tangents is the director circle of § 185.

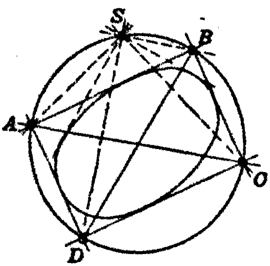


FIG. 105

The director circle of any conic is concentric with the given conic, and the square of its radius equals $a^2 \pm b^2$ where a and b are the semi-major and semi-minor axes of the conic.

187. Quadrangle Inscribed in a Circle and the Circumscribing Hyperbolas.

THEOREM. *The hyperbolas circumscribing a simple quadrangle whose vertices lie on a circle have parallel axes whose directions bisect the angles formed by the pairs of opposite sides of the quadrangle.*

If lines a_1 and a_2, b_1 and b_2, are drawn from any point, parallel to the pairs of opposite sides of the quadrangle, they determine a hyperbolic involution at that point and on the infinitely distant line of the plane. The line m bisecting the lesser angle between the rays a_1 and a_2 is perpendicular

to the line n bisecting the lesser angle between b_1 and b_2, and these two lines harmonically separate both a_1, a_2, and b_1, b_2 (§ 38). Hence m and n are the double rays of the involution determined by a_1, a_2, and b_1, b_2. They therefore bisect the angles between all pairs of conjugate rays of the involution.

Any hyperbola through the vertices of the inscribed quadrangle cuts the infinitely distant line in a pair of conjugate points of the involution determined by a_1, a_2, and b_1, b_2 (§ 176), and the asymptotes of the hyperbola pass through these points. Hence the lines m and n bisect the angles between the directions of the asymptotes of all hyperbolas through the vertices of the quadrangle. But the axes of a hyperbola bisect the angles between the asymptotes. Hence, the axes of all these hyperbolas have the same directions; namely, the directions of lines bisecting the angles between the pairs of opposite sides of the inscribed quadrangle.

EXERCISES

1. If correlated rays in a pencil of the first order make equal angles with a fixed ray, the pencil is in involution and is of the hyperbolic type. What are the double elements?

2. Given two pairs of conjugate elements of a hyperbolic involution on a straight line; find the double elements and the center of the involution.

3. Two projectively related pencils of rays or pencils of planes being given, how can they be brought into such position as to form an involution?

4. In an elliptic involution, find a pair of conjugate elements which will harmonically separate a given pair of conjugate elements. Is there more than one such pair?

5. Through a given point there can always be drawn one pair of rays at right angles which are polar conjugates with respect to a given conic. These bisect the angles between the tangents to the conic from the given point, if such can be drawn.

6. If an involution on a conic is projected from any point of the conic on the axis of that involution, show that the pairs of conjugate points on the axis are polar conjugates with respect to the conic.

7. The vertices of all triangles circumscribed to a conic whose bases lie on a fixed tangent to the conic and are bisected at the point of contact of that tangent, lie on that diameter of the conic which passes through the point of contact, and the line joining the points of contact of the sides of any one of these triangles is parallel to the base.

8. The vertices of all isosceles triangles whose bases lie on a fixed tangent to a parabola and whose sides touch the parabola lie on a fixed straight line.

9. Through a point P, a line PQ is drawn parallel to the polar of P relative to a given parabola. If P moves on a fixed straight line, show that PQ will envelop a second parabola.

10. A variable tangent to a parabola intersects a fixed tangent at a point P at which a normal to the variable tangent is drawn. Show that this normal envelops a parabola.

11. If two planes rotate about two fixed straight lines and are always parallel respectively to conjugate diameters of a given conic, they intersect in the rays of a regulus or of a cone of the second order on which the two fixed straight lines lie.

12. If circles are described on the three diagonals of a complete quadrilateral as diameters, and two of them intersect, the third will pass through their points of intersection. The sides of any right angle whose vertex is one of these points of intersection will be tangent to a conic inscribed in the quadrilateral.

CHAPTER XIV

FOCI AND FOCAL PROPERTIES OF CONICS

188. The Foci of a Conic. DEFINITION. A *focus* of a conic is a point of its plane in which all pairs of polar conjugate rays are at right angles.

If a given conic has a focus, it is clear from the definition that it cannot lie outside the conic; for from any point outside, two tangents may be drawn to the conic and each of these is conjugate to itself and to no other line through the point. Neither can a focus lie on the conic for every ray through a point on the conic is conjugate to the tangent at that point.

Moreover, if a focus exists, it must lie on an axis; in other words, the diameter through a focus is an axis, since an axis is the only diameter perpendicular to chords conjugate to it.

The line joining two foci must be a diameter, for this line is conjugate to a chord perpendicular to it at each focus and consequently it is the polar of an infinitely distant point. Since it is a diameter, it must be an axis. Therefore, if a conic has foci, they must all lie on one axis.

189. Focus of a Circle. The center of a circle satisfies the definition of a focus, all pairs of conjugate rays through it being at right angles. Moreover, the center is the only focus of a circle, for through any other point there is only one pair of conjugate chords at right angles; namely, the pair in which one of them is a diameter. The circle will consequently be excluded from the following discussions.

190. Determination of the Foci of a Conic. To discover if foci of a general conic exist, we may proceed in the following manner.

Given a conic of which the line a is an axis (Fig. 106). We may take on a any point P_1, not the center, and through P_1 draw any chord p_1 of the conic, oblique to a. Find p_2, the conjugate normal of p_1, intersecting a at P_2 and p_1 at S.

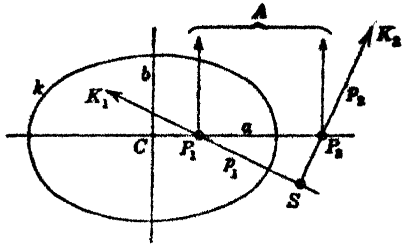


FIG. 106

The points P_1 and P_2 are non-conjugate and the pencils of rays having these points for centers are projectively related if to each ray of P_1 is correlated the conjugate ray through P_2 (§ 114).

If A is the infinitely distant pole of the axis a, the three rays, a, p_1, P_1A, through P_1 are perpendicular, respectively, to their conjugate rays P_2A, p_2, a, through P_2, and consequently, of the two pencils of rays P_1 and P_2, each ray through one of the points, is perpendicular to its conjugate ray through the other (§ 54).

By choosing different points P_1, Q_1, R_1, $\cdots$ on the axis a, and finding the corresponding points P_2, Q_2, R_2, $\cdots$ the points of the axis are correlated in pairs P_1, P_2; Q_1, Q_2; R_1, R_2, $\cdots$ such that the rays through one point of each pair are normal to the conjugate rays through the other point.

These pairs of points on the axis a form an involution; for if the points of the axis are projected from the infinitely distant point K_1 of p_1 and also from the infinitely distant point K_2 of p_2, and to each ray of the pencil K_1 is correlated its conjugate ray in K_2, the two pencils are projectively

related (§ 114) and the rays K_1P_1 and K_2P_2, also K_1P_2 and K_2P_1, are homologous. Hence the two superposed ranges of points on the axis a, sections of the pencils of parallel rays K_1 and K_2, are projectively related and the points P_1 and P_2 correspond doubly.

In particular, the ray K_1C of the pencil K_1, where C is the center of the conic, corresponds to the infinitely distant ray of K_2, for the pole of K_1C is infinitely distant and, joined to K_2, it gives the infinitely distant ray of K_2 as its conjugate. Similarly, the ray K_2C is homologous to the infinitely distant ray of K_1. Hence, in the involution $P_1, P_2; Q_1, Q_2; R_1, R_2, \cdots$ the center of the conic is homologous to the infinitely distant point of the axis a and is therefore the center of the involution.

If this involution on the axis a is hyperbolic, there are two points F_1 and F_2 on that axis, the double points of the involution, which satisfy the definition of a focus, for in each of them every ray is perpendicular to its conjugate ray. The two foci, F_1 and F_2, harmonically separate all pairs of points P_1, P_2 of the involution and are equally distant from the center.

In case the conic under consideration is a parabola and the point P_1 is the infinitely distant point of the axis a, the pole of any chord through P_1 is on the infinitely distant line, and the conjugate normal to such a chord is the infinitely distant line itself. Hence P_2 and P_1 coincide at the infinitely distant point of the parabola and this point is one of the double points of the involution on the axis; that is, the infinitely distant point on the axis is one of the foci of the parabola.

If the involution on the axis a is elliptic, as it may be for an ellipse or a hyperbola, and consequently has no double elements, there are two points, F_1 and F_2, on the axis b, normal to a through the center of the conic, from which

the involution on a is projected by orthogonal pencils of rays (§ 169). In this case, the two points, F_1 and F_2, on the axis b satisfy the definition of a focus, for in each of them conjugate rays are perpendicular to each other.

No points other than F_1 and F_2 on either axis of an ellipse or a hyperbola satisfy the definition of a focus; consequently we drawn the following conclusions.

An ellipse or a hyperbola has two actual foci. These lie on one axis and are equally distant from the center. A parabola has one actual focus, the second focus coinciding with the infinitely distant point of the conic.

The axis of an ellipse or hyperbola on which the foci lie is called the ***principal*** or ***major axis*** and the other is the ***conjugate*** or ***minor axis***. The major axis of a hyperbola intersects the curve, while the minor or conjugate axis does not. In an ellipse, both axes intersect the curve.

191. Construction for the Foci of an Ellipse or a Hyperbola. In the plane of an ellipse or a hyperbola, if two lines are drawn which are conjugate with respect to the conic and normal to each other, they will intersect the major axis of the conic in points harmonically separated by the foci and will intersect the minor axis in points which are projected from either focus by conjugate rays at right angles (§ 190).

Suppose, then, there is given an ellipse or a hyperbola and its major and minor axes, and it is required to find the foci.

A right triangle may be drawn whose hypotenuse lies on the minor axis and whose sides are conjugate with respect to the conic. The circle circumscribing this triangle will intersect the major axis at the foci.

For, if ABC is the triangle, C being the right angle (Fig. 107), and through A and B lines are drawn parallel to the major axis, through either of these points there are three rays conjugate and normal, respectively, to three rays through the other. Hence, all pairs of conjugate rays through

the two points are at right angles (§ 54), and, conversely, all pairs of rays through these points which are at right angles are conjugate. If F_1 and F_2 are the points in which the circle about ABC cuts the major axis, the pairs of conjugate rays through F_1, say, form an involution in which F_1A and F_1B are at right angles, as are also the axis and normal chord through F_1. Consequently, all pairs of conjugate rays through F_1 are at right angles (§ 170), and the same is true for F_2. The points F_1 and F_2 are therefore the foci of the conic.

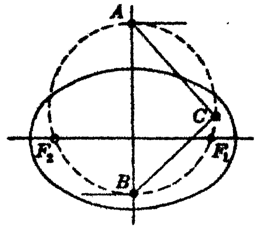


Fig. 107

192. Foci of an Ellipse. To find the foci of an ellipse whose major and minor axes are given, the following is a method simpler than that of § 191.

Draw tangents A_1B_1 and A_2B_2 at the extremities of the major axis (Fig. 108) and draw a tangent B_1B_2 at one extremity of the minor axis. These tangents form a circumscribed triangle about the ellipse whose vertices are B_1, B_2, and the infinitely distant intersection of the tangents at A_1 and A_2. Any point, therefore, on A_1A_2 is projected from B_1 and B_2 by conjugate rays (§ 115). If a circle is described on B_1B_2 as diameter, it will intersect the major axis in points F_1 and F_2 in which pairs of conjugate rays are at right angles. Hence F_1 and F_2 are the foci of the ellipse. (§ 170.)

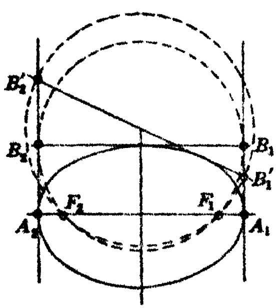


Fig. 108

It may be added that a circle whose diameter is the segment of any other tangent to the conic intercepted between

the tangents at A_1 and A_2 will likewise intersect the major axis at the foci. This method of finding the foci of an ellipse was known to Apollonius.[1]

193. Foci of a Hyperbola. A similar construction for the foci of a hyperbola whose major axis and asymptotes are given is the following. Let the tangents at the extremities of the major axis, A_1 and A_2 (Fig. 109), intersect one asymptote at B_1 and B_2, respectively. These tangents and the asymptote form a circumscribed triangle and any point of A_1A_2 is projected from B_1 and B_2 by conjugate rays (§ 115). If a circle is drawn on B_1B_2 as diameter, it will cut the major axis in points F_1 and F_2 which are projected from B_1 and B_2 by conjugate rays at right angles. As in the case of the ellipse, these points are the foci.

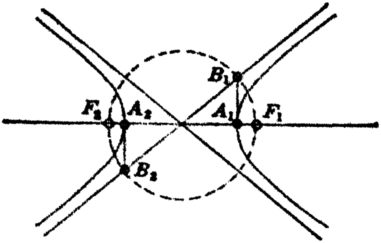


FIG. 109

194. Focus of a Parabola. In § 190 it was pointed out that for any conic, the intersections of a pair of conjugate normal rays with the major axis are harmonically separated by the foci of the conic, and since for a parabola one focus is infinitely distant, the segment of the axis intercepted between any pair of conjugate normal rays is bisected at the focus.

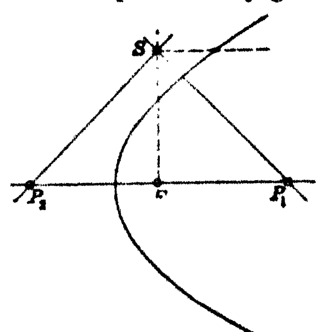


FIG. 110

Hence, to find the focus of a parabola whose axis is given,

[1] Apollonius, *Conicorum*, Bk. III.

we need only to construct a pair of conjugate normal rays (Fig. 110) intersecting the axis at the points P_1 and P_2, and the mid-point of the segment P_1P_2 is the focus F.

195. The Tangent of a Conic makes Equal Angles with the Focal Radii of its Point of Contact. If any point S in the plane of a conic is joined to the foci F_1 and F_2, the rays SF_1 and SF_2 are called the *focal radii* of the point. The conjugate normal rays through S relative to the conic and the focal radii of the point S, therefore, are harmonic, (§ 190), and the angles made by the focal radii are bisected by the conjugate normals (§ 38). If S is a point of the conic one of the conjugate normals becomes the tangent at S while the other is normal to the conic. This property, then, takes the following form.

THEOREM. *The tangent at any point of an ellipse or a hyperbola makes equal angles with the focal radii of the point, and the normal to the conic at that point bisects the angle between the focal radii.*

For a parabola, since one focus is infinitely distant, the theorem may be stated as follows.

THEOREM. *The tangent at any point of a parabola makes equal angles with the focal radius and the diameter through the point, and the normal to the curve bisects the angle between these lines.*

196. Confocal Conics intersect at Right Angles. If two conics have the same foci they are said to be *confocal*. When confocal conics intersect (Fig. 111), the focal radii of a common point make equal angles at that point with the tangents and with the normals to both conics (§ 195). The tangent to one of the intersecting conics

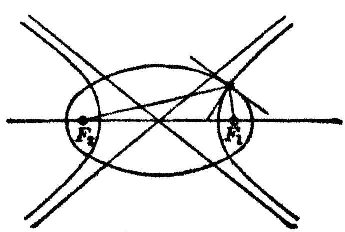


FIG. 111

must therefore be the normal to the other and we have the following theorem.

THEOREM. *Confocal conics which intersect cut one another orthogonally.*

197. Tangents to a Conic make Equal Angles with the Focal Radii of their Point of Intersection. If S is the point of intersection of two tangents to a conic, conjugate lines through S (Fig. 112) and the two tangents from S are harmonic (§ 107); hence, the conjugate normal lines through S bisect the angles made by the tangents. But conjugate normals also bisect the angles made by the focal radii of S (§ 195). Consequently, the following theorem may be stated.

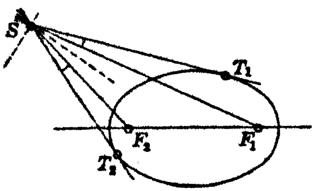


FIG. 112

THEOREM. *The tangents to a conic from any point make equal angles with the focal radii of that point.*

For a parabola this theorem takes the following form.

THEOREM. *If from any point two tangents are drawn to a parabola, one of them makes the same angle with the focal radius of the point as the other makes with the diameter through the point.*

A particular case of this theorem arises if the point S from which the two tangents are drawn lies on the tangent at the vertex of the parabola. If A is the vertex of the parabola, F, the focus (Fig. 113), and P, the point of contact of the second tangent from S, the angle FSA equals the angle KSP, where SK has the direction of the axis. Therefore the angle FSP equals KSA, a right angle. From this we have the following interesting relation.

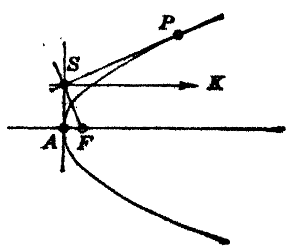


FIG. 113

THEOREM. *A line drawn from the focus of a parabola perpendicular to a tangent intersects it on the tangent at the vertex.*

198. Tangent and Normal in Relation to the Foci. The tangent and normal at any point of a conic are conjugate lines at right angles. Their intersections with the major axis, therefore, are harmonically separated by the foci and their intersections with the minor axis are projected from either focus by rays at right angles (§ 190). If then a circle is drawn through the foci and any point of a central conic (Fig. 114), its intercept on the minor axis is a diameter of the circle, and the extremities of this diameter lie on the tangent and normal, respectively, of the point.

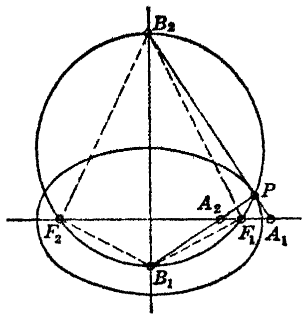


FIG. 114

The tangent and normal at any point of a parabola intercept a segment on the axis which is bisected at the focus.

199. The Focus of a Parabola is on the Circumscribing Circle of a Tangent Triangle. From the theorem of § 197 as applied to a parabola, the following property is readily deduced.

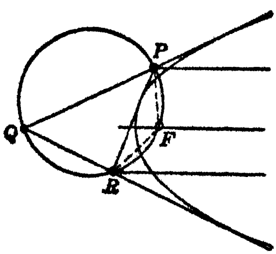


FIG. 115

THEOREM. *If the sides of a triangle are tangent to a parabola, the circle through its vertices passes through the focus of the parabola.*

If P, Q, R, are the vertices of the given triangle (Fig. 115) and F is the focus of the parabola, the angle which PQ makes with

PF equals the angle which PR makes with the axis of the parabola (§ 197). Also, the angle which RQ makes with RF equals the angle which RP makes with the axis.

Therefore the angle FPQ equals, or is supplementary to, the angle FRQ, and the points P, Q, R, F are on the same circle.

Since this circle is determined by the points P, Q, R, it follows that the foci of all parabolas touching three given lines lie on a circle through the intersections of these lines.

200. Construction for the Focus of a Parabola. To find the focus of a parabola of which four tangents, a, b, c, d, are given, if a, b, c determine the triangle PQR, and b, c, d determine the triangle RST, the circles circumscribing PQR and RST will intersect in R and in the focus.

From this property we have the following theorem.

THEOREM. *The circles circumscribing the four triangles formed by four given lines pass through one point.*

201. Properties of the Directrix of a Conic. DEFINITION. In any conic the polar of a focus is called a ***directrix*** of the conic.

An ellipse or a hyperbola has two foci; consequently, it has two directrices. A parabola, however, has only one actual directrix. Since a focus always lies on an axis and inside the conic, a directrix is perpendicular to that axis and lies wholly outside the conic. For a parabola, the segment of the axis intercepted between the focus and the directrix is bisected by the curve.

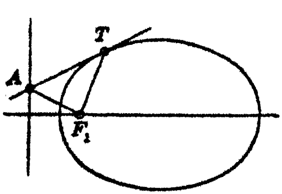


FIG. 116

From the definition of a directrix the following theorem becomes at once evident.

THEOREM. *The intercept on any tangent to a conic between its point of contact and a directrix subtends a right angle at the corresponding focus.*

If the tangent at a point T intersects a directrix at A (Fig. 116), the pole of this directrix being F_1, the line AF_1 is conjugate to TF_1 and hence the angle AF_1T is a right angle.

202. Tangents to a Parabola from a Point on the Directrix.

THEOREM. *The tangents to a parabola drawn from any point of the directrix are at right angles to each other* (see § 172).

If P is any point on the directrix of a parabola (Fig. 117) and tangents from it meet the parabola at A and B, the chord AB passes through the focus F (§ 105). The angle PAF equals the angle APC where PC has the direction of the diameters of the parabola (§ 197); also, the angle PBF equals the angle BPC; therefore, the angle APB is a right angle.

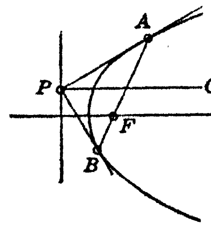


FIG. 117

203. The Tangents from a Point Subtend Equal Angles at a Focus.

THEOREM. *If a focus of a conic is joined to the points of contact of two tangents and to their point of intersection, the latter line makes equal angles with the two former.*

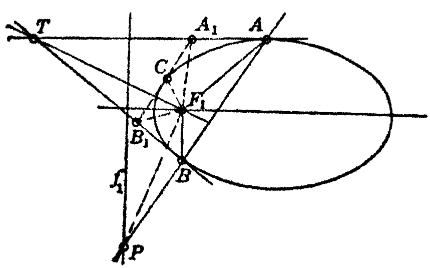


FIG. 118

Let tangents to a conic at points A and B (Fig. 118) intersect at the point T, and let F_1 be one focus of the

conic and f_1 be the corresponding directrix. If the chord AB intersects the directrix at P, the line TF_1 is the polar of P (§ 105), and the lines TF_1 and PF_1 are conjugate rays intersecting at a focus. The angle TF_1P is therefore a right angle.

Also, P and TF_1 harmonically separate the points A and B, and hence the rays $F_1(A, T, B, P)$ are harmonic. Therefore F_1T makes equal angles with F_1A and F_1B (§ 38). In other words, the tangents TA and TB subtend equal angles at the focus F_1.

204. Two Tangents to a Conic are cut by a Variable Tangent in Points Projected from a Focus by Equal Pencils. To a conic of which F_1 is a focus, if there are given two fixed tangents meeting it at the points A and B, respectively (Fig. 118), and if a variable tangent meets the conic at the point C and intersects the fixed tangents at A_1 and B_1, respectively, the angle A_1F_1C equals one-half the angle AF_1C (§ 203) and the angle B_1F_1C equals one-half the angle BF_1C. Therefore the angle $A_1F_1B_1$ equals one-half the angle formed by the fixed lines F_1A and F_1B, and is constant. Hence the ranges of points determined on the fixed tangents at A and B by the variable tangent at C are projected from F_1 by rays making a constant angle with each other. We may therefore announce the following theorem.

THEOREM. *The ranges of points in which two fixed tangents to a given conic are cut by a variable tangent are projected from a focus by equal and directly projective pencils of rays.*

From this property it follows that if three tangents and a focus of a conic are given, any required number of tangents can be readily found.

205. Triangle Circumscribed to a Parabola. If the given conic of § 204 is a parabola (Fig. 119), the variable tangent in one position may be the infinitely distant line of the plane

and the constant angle is the supplement of the angle made by the two fixed tangents. A circle may, therefore, be circumscribed to the quadrangle whose sides are the two fixed tangents and the focal rays FA_1 and FB_1. In this we have another proof of the property that the circle through the vertices of a triangle circumscribed to a parabola passes through the focus (§ 199).

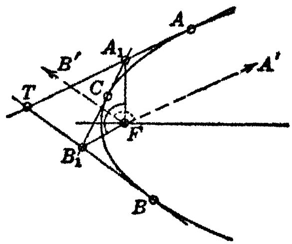


FIG. 119

206. Triangle Circumscribed to a Hyperbola. If the given conic in § 204 is a hyperbola and the two fixed tangents are the asymptotes, we have the following theorem.

THEOREM. *The two foci of a hyperbola and the points in which an arbitrary tangent intersects the asymptotes lie on a circle.*

If the foci are F_1 and F_2 and the points in which the arbitrary tangent intersects the asymptotes are A_1 and A_2 (Fig. 120), the angle $A_1F_1A_2$ is constant as is also the angle $A_1F_2A_2$ (§ 204).

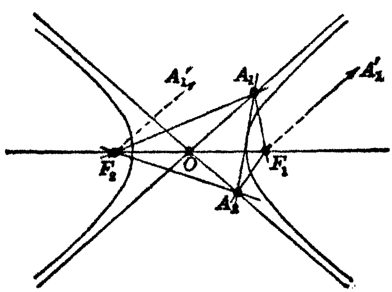


FIG. 120

If the variable tangent takes the position of the asymptote on which the point A_1 lies, then F_1A_1 becomes parallel

to that asymptote while F_1A_2, also F_2A_2, coincides with the axis F_1F_2. The constant angle $A_1F_1A_2$, therefore, is equal to one of the angles which an asymptote makes with the major axis and the constant angle $A_1F_2A_2$ is equal to the other angle which the same asymptote makes with the major axis. The angles $A_1F_1A_2$ and $A_1F_2A_2$ are, consequently, supplementary and a circle may pass through the points $F_1A_1F_2A_2$.

If O is the center of the given hyperbola, the angle A_1OA_2 is double the angle A_1OF_1 and therefore double the angle $A_1F_2A_2$; and if C is the center of the circle through A_1, A_2, and the two foci, the angle A_1CA_2 is likewise double the angle $A_1F_2A_2$. Hence the angles A_1OA_2 and A_1CA_2 are equal and a circle may pass through the points A_1, A_2, O, C. If A_1A_2 is tangent to the hyperbola at the vertex, O and C coincide, and the angle $F_1A_1F_2$ is a right angle, as is also $F_1A_2F_2$.

207. The Eccentricity of a Conic is Constant. The theorem of § 203 lends itself to the demonstration of the following important metric property of conics.

THEOREM. *The distance of a variable point of a given conic from a focus bears a constant ratio to its distance from the corresponding directrix.*

In Fig. 118 project the points A and B of the conic by lines parallel to TF_1, on the corresponding directrix f_1, in the points A_1 and B_1 (Fig. 121). Then A_1 and B_1 are harmonically separated by the point P and the line TF_1, and the rays F_1A_1 and F_1B_1 are harmonically separated by F_1P and F_1T.

Since TF_1P is a right angle, the angle A_1F_1T equals the angle B_1F_1T. But the angle A_1F_1T equals the angle F_1A_1A, and the angle B_1F_1T equals the angle F_1B_1B. Hence the angle F_1A_1A equals the angle F_1B_1B.

Also, the angle AF_1T equals the angle BF_1T (§ 203).

Hence, by subtraction, the angle A_1F_1A equals the angle B_1F_1B, and the triangles A_1F_1A and B_1F_1B are similar. Therefore

$$\frac{AF_1}{AA_1} = \frac{BF_1}{BB_1}.$$

But AA_1 and BB_1 are proportional to the perpendicular

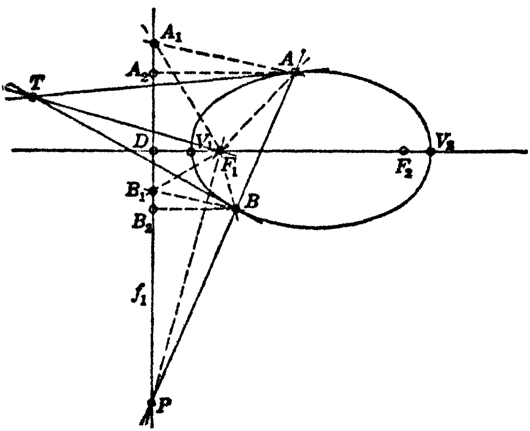


FIG. 121

distances AA_2 and BB_2 of A and B, respectively, from the directrix. Hence we have

$$\frac{AF_1}{AA_2} = \frac{BF_1}{BB_2}.$$

Since A and B are any points whatsoever of the conic, this ratio is constant for all points of the curve. The constant ratio AF_1/AA_2 is called the *eccentricity* of the conic and is commonly designated by the letter e.

208. Eccentricity of a Parabola. In a parabola the constant ratio AF_1/AA_2 is equal to unity since when A is at the vertex it is equally distant from the focus and the directrix

(§ 201). From this property the following definition may be stated.

Definition. A parabola is the locus of a point which moves in a plane so that its distance from a fixed point is equal to its distance from a fixed line.

209. Eccentricity of an Ellipse or a Hyperbola. In an ellipse or hyperbola, let the major axis intersect the curve at the vertices V_1 and V_2 and intersect the directrix corresponding to the focus F_1 at the point D_1. Then the constant ratio AF_1/AA_2 (§ 207) equals V_1F_1/V_1D_1. But V_1 and V_2 are harmonically separated by F_1 and D_1 (§ 103). Therefore, considering only the magnitude of the segments, V_1F_1/V_1D_1 equals V_2F_1/V_2D_1.

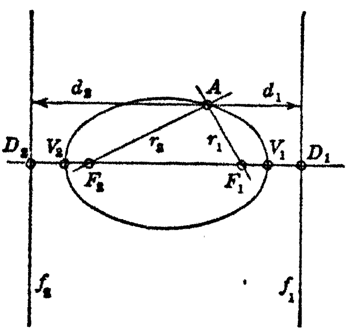


Fig. 122

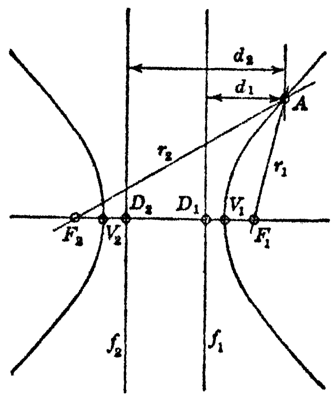


Fig. 123

In an ellipse (Fig. 122) the ratio V_2F_1/V_2D_1 is less than unity, while in a hyperbola (Fig. 123) this ratio is greater than unity. As in the case of the parabola, these properties may be taken as bases of definitions of the two central conics in the following form.

Definition. An ellipse is the locus of a point which moves in a plane so that the ratio of its distance from a

fixed point to its distance from a fixed line is constant and less than unity.

DEFINITION. A hyperbola is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point to its distance from a fixed line is constant and greater than unity.

These metric properties of conics may be summarized briefly by saying that the eccentricity of a parabola is unity; of an ellipse, it is less than unity; and of a hyperbola, it is greater than unity.

210. Other Metric Properties of Central Conics. An ellipse or a hyperbola is symmetric with respect to either axis and consequently the ratio AF_1/AA_2 (§ 207) is the same whether one focus and the corresponding directrix is used or the other.

Suppose r_1 and r_2 (Figs. 122, 123) are the distances of any point A of an ellipse or a hyperbola from the foci, F_1 and F_2, respectively, and that d_1 and d_2 are the distances of this point from the corresponding directrices. Then, in an ellipse, $d_1 + d_2$ is the distance between the directrices and in a hyperbola, $d_1 - d_2$, or its negative, is that distance. For either curve the distance is constant.

For either curve, also, $r_1/d_1 = r_2/d_2 = e$, a constant for all points of the curve (§ 207). From this property we obtain the following relation.

$$\frac{r_1 + r_2}{d_1 + d_2} = \frac{r_1 - r_2}{d_1 - d_2} = \text{a constant.}$$

In an ellipse, $d_1 + d_2$ is constant; hence $r_1 + r_2$ is constant.

In a hyperbola, $d_1 - d_2$ is constant; hence $r_1 - r_2$ is constant. From these relations we have the following theorem.

THEOREM. *In an ellipse the sum of the distances from any point of the curve to the two foci is constant, while in a hyperbola the difference of those distances is constant.*

These properties, again, frequently serve as metric definitions of the central conics. In either curve this constant sum or difference ($r_1 \pm r_2$) is equal to the length ($2\,a$) of the major axis, as readily appears if A is taken at a vertex on that axis. Moreover, in an ellipse, the distance from an extremity of the minor axis to a focus equals a, the semi-major axis, and it follows that a is greater than b, the length of the semi-minor axis. No such conclusion can be drawn regarding the axes of a hyperbola.

211. Locus of Points Inverse to a Focus relative to a Variable Tangent. The inverse of any point with respect to a given line is found by drawing the perpendicular from the point to the line and producing it an equal distance on the other side. With this definition, the following theorem may be stated.

THEOREM. *The locus of points inverse to a focus of an ellipse or a hyperbola with respect to the tangents of the curve is a circle whose center is the other focus and whose radius is equal to the major axis of the conic.*

In Fig. 124, if A is any point of an ellipse (or a hyperbola) whose foci are F_1 and F_2, and AM is the tangent at A, since F_2A and F_1A make equal angles with the tangent (§ 195), F_2A will intersect the perpendicular from F_1 on the tangent, at a point F inverse to F_1. Also, F_2F equals $2\,a$, the sum (or difference) of F_2A and F_1A (§ 210). Hence, the locus of F, as A moves along the conic, is a circle whose center is F_2 and whose radius equals $2\,a$.

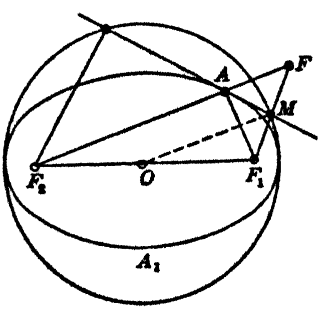


FIG. 124

212. The Auxiliary Circle of a Central Conic. In Fig. 124, the point M is the foot of the perpendicular from the focus F_1 on the tangent at A and is the midpoint of F_1F. If M is joined to the center O of the conic, MO is equal to one-half of F_2F, and is therefore constant and equal to a. From this property we have at once the following theorem.

THEOREM. *The locus of the foot of the perpendicular from a focus on a tangent of an ellipse or a hyperbola is a circle on the major axis as diameter.*[1]

This circle is called the ***auxiliary circle*** of the conic. The converse of the preceding theorem may be stated as follows.

THEOREM. *If a right angle moves in its plane in such a way that its vertex describes a fixed circle while one side passes constantly through a fixed point, the other side will envelop a conic concentric with the given circle, of which the fixed point is one focus.*

The conic is an ellipse or a hyperbola according as the given point lies within or without the given circle.

The corresponding theorem for a parabola which is converse to the property proved in § 197 may be stated as follows.

THEOREM. *If a right angle moves in its plane in such a way that its vertex describes a straight line while one side passes through a fixed point, the other side will envelop a parabola of which the fixed point is the focus and the straight line is the tangent at the vertex.*

213. Properties of the Latus Rectum of a Conic. DEFINITION. The chord of a conic through a focus perpendicular to the axis is called the ***latus rectum*** of the conic. This chord is also called the ***parameter*** of the conic.

An interesting metric property of the latus rectum is stated in the following theorem.

[1] Apollonius, *Conicorum*, Bk. III, pp. 49 and 50.

THEOREM. *The semi-latus rectum of a conic is a harmonic mean between the segments determined by the focus on any focal chord.*

Suppose M_1M_2 is any chord of a given conic passing through a focus (Fig. 125), of which F_1L is the semi-latus

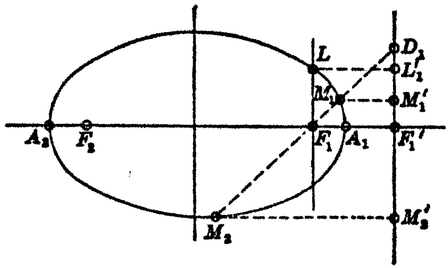


FIG. 125

rectum. Let M_1M_2 intersect the corresponding directrix at the point D_1. Since the directrix is the polar of the focus, the points D_1, M_1, F_1, M_2, are harmonic, from which it follows (§ 37) that

$$\frac{2}{D_1F_1} = \frac{1}{D_1M_1} + \frac{1}{D_1M_2}.$$

If perpendiculars M_1M_1', F_1F_1', M_2M_2', are drawn to the directrix, these are proportional to the segments D_1M_1, D_1F_1, D_1M_2, respectively, and hence

$$\frac{2}{F_1F_1'} = \frac{1}{M_1M_1'} + \frac{1}{M_2M_2'}.$$

But we know that

$$\frac{M_1F_1}{M_1M_1'} = \frac{M_2F_1}{M_2M_2'} = \frac{LF_1}{F_1F_1'} = e,$$

the eccentricity of the conic (§ 207). Therefore

$$\frac{2}{LF_1} = \frac{1}{M_1F_1} + \frac{1}{M_2F_1}.$$

This is to say, LF_1 is the harmonic mean between M_1F_1 and M_2F_1.

In particular, if the focal chord is the major axis, we have the relation

$$\frac{2}{LF_1} = \frac{1}{A_1F_1} + \frac{1}{A_2F_1}.$$

That is to say, the semi-latus rectum is the harmonic mean between the distances from a focus to the extremities of the major axis.

For a parabola, one of the focal distances A_2F_1 is infinite and we have the relation, $LF_1 = 2A_1F_1$.

For an ellipse or a hyperbola, denoting the semi-latus rectum by l and the focal distances of the vertices by $a + c$ and $a - c$, where $a^2 - c^2 = \pm b^2$, a and b being the semi-axes of the conic, and using the positive or negative sign according as the conic is an ellipse or a hyperbola, the relation just written takes the form

$$\frac{2}{l} = \frac{1}{a - c} + \frac{1}{a + c} = \frac{2a}{b^2}.$$

Therefore, considering magnitude only, for either of these curves, $l/b = b/a$, or the minor axis is a mean proportional between the major axis and the latus rectum.

214. Perpendiculars from the Foci on a Tangent. In an ellipse or a hyperbola (Figs. 126, 127), O being the center of

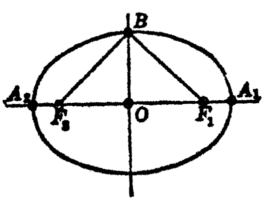


FIG. 126

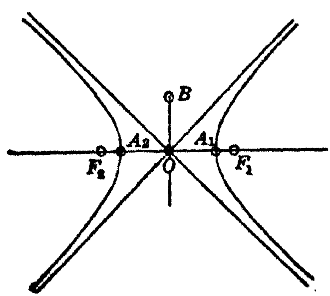


FIG. 127

the conic, we have the relations

$$F_1A_1 = F_1O + OA_1 = OA_1 - OF_1.$$
$$F_1A_2 = F_1O + OA_2 = -(OA_1 + OF_1).$$

Therefore

$$F_1A_1 \cdot F_1A_2 = -(OA_1^2 - OF_1^2) = \mp OB^2,$$

which is a constant, the positive or negative sign being used according as the conic is a hyperbola or an ellipse. Moreover,

$$F_1A_1 \cdot F_1A_2 = -A_1F_1 \cdot A_1F_2 = \mp OB^2,$$

which is a constant. If, now, M_1M_2 and $M_1'M_2'$ (Fig. 128) are parallel tangents of an ellipse or a hyperbola and F_1M_1 and F_1M_1' are perpendiculars from F_1 on these tangents, M_1 and M_1' will lie on the auxiliary circle of the conic (§ 212); and consequently

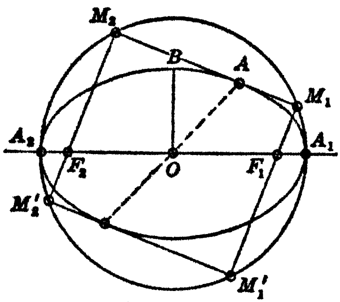


FIG. 128

$$F_1M_1 \cdot F_1M_1' = F_1A_1 \cdot F_1A_2 = \pm OB^2,$$

which is a constant. From this we have the following theorem.

THEOREM. *The product of the perpendiculars from a focus of an ellipse or a hyperbola on parallel tangents is constant and equal to the square on the minor axis.*

If F_2M_2 and F_2M_2' are the perpendiculars from the other focus F_2 on these parallel tangents, F_2M_2 equals F_1M_1' and we have the following theorem.

THEOREM. *The product of the perpendiculars from the foci of an ellipse or a hyperbola on a variable tangent is constant and equal to the square on the minor axis.*

EXERCISES

1. Construct a conic having given one focus, the corresponding directrix, and one point or one tangent.

2. Construct a conic having given the two foci and one point or one tangent.

3. Construct a parabola having given the focus and the directrix, or the focus, the direction of the diameters, and one point or one tangent.

4. The poles of a given straight line with respect to a system of confocal conics lie on a straight line perpendicular to the given line and harmonically separated from it by the two foci.

The proof of the relation here stated leads readily to the following:

a) In a system of confocal conics there is one and only one conic which is tangent to a given line.

b) Through any point of the plane of a system of confocal conics two conics of the system pass and these intersect at right angles.

5. The angles made by the two tangents from a given point to the several conics of a confocal system are all bisected by two fixed straight lines; namely, by the tangents to the two conics of the system passing through the given point.

6. Each pair of opposite sides of a quadrangle determined by the vertices of a self-polar triangle of a given conic and its orthocenter, is harmonically separated by the foci of the conic.

7. The centers of all circles tangent to both of two given circles lie on two confocal conics of which the centers of the given circles are the foci.

8. All points of a plane which are equally distant from a circle and a straight line lie on one or the other of two parabolas of which the center of the circle is the focus.

9. If from any point of a circle circumscribing a triangle, perpendiculars are drawn to the sides of the triangle, they will intersect the sides in points of a straight line (Simson's Line or the Pedal Line).

10. If one side of an angle of any given magnitude passes through one focus of a given conic, while the other side is tangent to the conic, the vertex of the angle describes a circle which touches the conic at two points; if the given conic is a parabola, the vertex describes a straight line tangent to the parabola.

CHAPTER XV

IMAGINARY ELEMENTS—PROBLEMS OF THE SECOND ORDER

215. Distinction between Real and Imaginary Elements. In Chapter XIII, it was pointed out that an involution in an elementary form is fully determined when two pairs of conjugate elements are known.

If A_1, A_2, and B_1, B_2, for example, are two pairs of conjugate points in an involution on a straight line, it is always possible from them to determine a point C_2 conjugate to any point C_1 of the line, such that, among other relations, we may write

$$A_1B_1A_2C_1 \barwedge A_2B_2A_1C_2,$$

and

$$A_1B_2B_1C_1 \barwedge A_2B_1B_2C_2.$$

The method employed for such determination was, first, to project the two given pairs of points, A_1, A_2, and B_1, B_2, from any center on a conic passing through that center; then the lines joining the pairs of projected points on the conic, $A_1'A_2'$ and $B_1'B_2'$, intersect in a point U, the so-called center of the involution on the conic, and the lines $A_1'B_2'$, $A_2'B_1'$, and $A_1'B_1'$, $A_2'B_2'$ intersect in points of the so-called axis of that involution. Pairs of conjugate points in the involution on the conic, C_1', C_2', lie in a straight line with the center U, and when one of them is given the other is determined. By projecting the pairs of conjugate points in the involution on the conic, from the same center, back on the given line, the involution on the line is determined.

The center and the axis of the involution on the conic can always be found no matter how the points A_1, A_2; B_1, B_2,

on the given line, and their corresponding points on the conic, are placed in order, and it is only when a question is raised as to the existence of self-conjugate or double points in the involution that consideration needs to be given to their arrangement.

If the pairs of conjugate points A_1, A_2, and B_1, B_2, on the given line, and likewise the corresponding points on the conic, do not separate each other (Fig. 129), the center of the involution on the conic lies outside the conic and the tangents from it to the conic determine two points which are

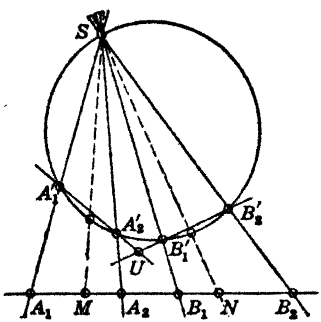


FIG. 129

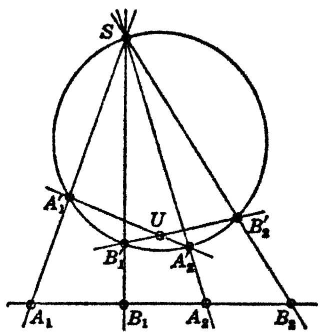


FIG. 130

conjugate to themselves, the so-called double points of the involution. When projected back on the straight line these double points give rise to points, which are self-conjugate in the involution on the line. If, on the other hand, the pairs of points A_1, A_2; B_1, B_2, on the given line, and on the conic, separate each other (Fig. 130), the center of the involution lies inside the conic and no double elements of the involution are found.

For purposes of generality we speak of the involution as having double elements in both cases. In the first case, they are *real* and can be found; in the second, they cannot be found and are said to be *imaginary*. In other words, the

double elements in a hyperbolic involution are real and in an elliptic involution they are imaginary.

216. Illustrations of Imaginary Elements. No attempts are made in pure geometry to construct imaginary elements or to represent them by diagrams. Their consideration grows out of the acceptance of the principle of continuity and a desire for generality of expression. For example, we may make the following statements.

1. Two superposed projective forms of the first order have two self-corresponding elements, real or imaginary.
2. Two pairs of points on a straight line are harmonically separated by the same third pair, real or imaginary, according as the two given pairs do not or do separate each other.
3. In an involution on a straight line determined by two pairs of conjugate points, there are always two double elements. These elements are real or imaginary according as the two chosen pairs of points do not or do separate each other.
4. Among the pairs of points on a straight line which are polar conjugates with respect to a given conic, there are two points in which conjugate pairs coincide. These are real if the line intersects the given conic and imaginary if it does not.
5. Every line in the plane of a conic intersects the conic in two points, real or imaginary.
6. Through any point of a plane two tangents to a conic may be drawn. The tangents are real if the point lies outside the conic, and imaginary if it lies inside.
7. Through any line passing through the vertex of a cone of the second order, two planes intersect which are tangent to the cone. These planes are real if the line lies outside the cone, and imaginary if the line lies inside.
8. Any plane through the vertex of a cone of the second order contains two rays of the cone, real or imaginary.

In the above statements, the two real elements or the two imaginary elements may coincide in a single real element. The two points of intersection of a straight line and a conic, for instance, coincide if the line is tangent to the conic; and the two points on a straight line which are polar self-conjugates relative to a conic coincide under the same condition.

Imaginary elements always occur in pairs, the two being spoken of as ***conjugate imaginaries.*** Whenever one imaginary element arises, there is a conjugate imaginary.

Moreover, from what has been said it will readily appear that two conjugate imaginary points always lie on a real line and two conjugate imaginary lines in a plane intersect in a real point. Two conjugate imaginary planes also intersect in a real line.

217. Imaginary Elements of the Second Kind. An involution may be established among the rays of a regulus of the second order by setting up an involution among the points of a plane section of the regulus and correlating the rays of the regulus to the points of the section through which they pass. If the double elements of the involution on the section are real or imaginary, so also will be the double elements of the involution in the regulus. Thus, a pair of conjugate imaginary rays may arise which neither intersect in a real point nor lie in a real plane. Such rays are said to be conjugate imaginary rays of the second kind, to distinguish them from conjugate imaginary rays which lie in a real plane and intersect in a real point.

218. Properties of Imaginary Elements. The addition of imaginary elements to the totality of elements heretofore considered; namely, real points, lines, and planes of space, including the ideal points and lines of a single plane at infinity, adds greatly to the difficulty of geometric study by projective methods. It is to Von Staudt that credit is due

for having founded the purely geometric theory of imaginary elements and for having established the fact that such elements, arising as defined in the preceding paragraphs, obey the same laws of combination as do real elements.

A first difficulty met with in the study of geometric imaginaries is a means of distinguishing between the elements of a conjugate pair which cannot be located or visualized. This Von Staudt accomplished by adding the notion of sense or order to the elements of the involution defining a pair of imaginaries. Thus, if the pairs of points A_1, A_2; B_1, B_2, in an elliptic involution define a pair of conjugate imaginary points, one of them may be designated $A_1B_1A_2$ and the other $A_2B_1A_1$.

For our purposes, however, it will be sufficient to deal with imaginary elements in pairs of conjugates, and their consideration as separate elements need not be extended. The following brief summary of properties of imaginaries may be given.

1. Through an imaginary point there passes one and only one real line; namely, the line on which its conjugate imaginary lies.

2. In an imaginary plane there lies one and only one real line; namely, the line of intersection with the conjugate imaginary plane.

3. On an imaginary line of the first kind there lies one and only one real point; namely, the point of intersection with the conjugate imaginary line.

4. On an imaginary line of the second kind there lies no real point and through it there passes no real plane. It neither intersects nor lies in a plane with its conjugate imaginary.

219. Problems of the Second Order. Problems which admit of two solutions are spoken of as problems of the second order; and the solutions may be real and distinct, or

they may coincide, or they may be conjugate imaginary solutions. The following will serve as a first illustration.

PROBLEM. *Given five points, S_1, S_2, A, B, C, of a conic, to find where the conic intersects a given straight line.*

Since a conic is the locus of intersections of pairs of homologous rays in two projective pencils lying in the same plane, the problem reduces to that of finding the points of the given line in which a pair of homologous rays of the projective pencils $S_1(A, B, C, \cdots)$ and $S_2(A, B, C, \cdots)$ intersect; that is, to finding the self-corresponding points in the two projective ranges $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$ determined on the given line by the pencils of rays $S_1(A, B, C, \cdots)$ and $S_2(A, B, C, \cdots)$. If these ranges of points are projected from any center S on a conic k passing through S (Fig. 131), we have a projectivity on k whose axis is

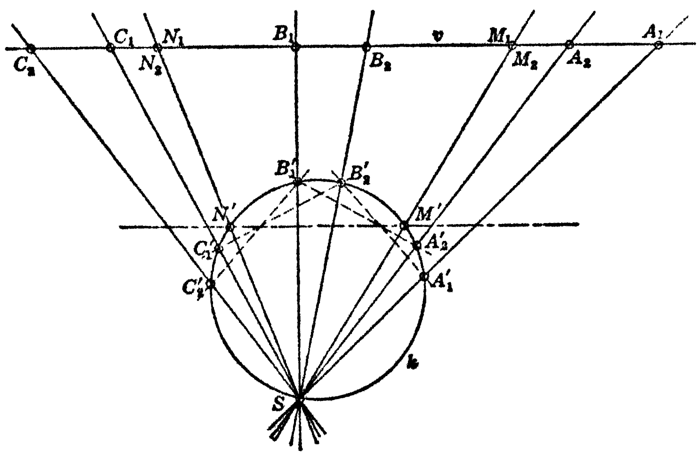


FIG. 131

readily determined (§ 154). If the axis of projectivity intersects the conic k in two real points, M' and N', these intersections are self-corresponding points of the projectivity

on the conic and when projected from S back on the given line, they determine two real self-corresponding points of the ranges $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$, and in them pairs of homologous rays of the pencils S_1 and S_2 intersect. They are thus the points which the conic determined by S_1, S_2, A, B, C, has in common with the given line.

If the axis of the projectivity on the conic k is tangent to k, the points M' and N' coincide, as do also the intersections of the given conic with the given line; and if this axis intersects k in imaginary points, that is, if the axis does not intersect k in real points, nor is tangent to it, the conic intersects the given line in imaginary points.

The reciprocal problem, which is solved in a wholly analogous way, would read as follows.

PROBLEM. *Given five tangents of a conic, to draw the tangents which pass through a given point.*

Other illustrations of problems of the second order follow.

220. Conics through Four Points which are Tangent to a Given Line.

PROBLEM. *To construct a conic through the vertices of a simple quadrangle which is tangent to a given line.*

By Desargues' theorem (§ 176) a conic through the vertices of the given quadrangle passes also through a pair of conjugate points of the involution determined on the given line by the pairs of opposite sides of the quadrangle. Since a pair of conjugate points of the involution coincides at a double point, the conic through the vertices of the quadrangle and a double point of the involution is tangent to the given line at that point. Hence, the problem reduces to that of finding the double points in the involution on the given line determined by the sides of the quadrangle and constructing the conic through one of these points and the four vertices of the quadrangle.

Since, in general, there are two double points in the involution, two conics may be constructed which satisfy the given conditions. If, however, the line is so situated relative to the vertices of the quadrangle that the involution on it determined by the sides of the quadrangle is elliptic, no conic can be constructed which will satisfy the given conditions.

221. Construction of a Triangle Inscribed to one Given Triangle and Circumscribed to another.

PROBLEM. *Given two triangles in the same plane, it is required to construct a third triangle whose sides pass through the vertices of one of the given triangles and whose vertices lie on the sides of the other.*

In other words, we are required to construct a triangle which is inscribed to one and circumscribed to the other of two given triangles.

If ABC and PQR are the given triangles (Fig. 132), we may require that the sides of the triangle to be constructed shall pass through A, B, C, respectively, of the one triangle and that its vertices shall lie on the sides p, q, r, respectively, of the other.

From A project the points P_1, P_2, P_3, $\cdots$ of the side p into the points Q_1, Q_2, Q_3, $\cdots$ on the side q; from B project the points Q_1, Q_2, Q_3, $\cdots$ of q, perspective to those of p, on the side r; and from C project the points of r in turn back into the points P_1', P_2', P_3', $\cdots$ on the side p. On the side p there are thus two projective ranges of points, p and p', in which there are, in general, two self-corresponding points, K_1 and K_2, and these may be readily determined.

If K_1 is projected from A into L_1 on r, and L_1 is projected from B into M_1 on q, M_1 will be projected from C back into K_1 on p.

Thus $K_1L_1M_1$ is a triangle whose sides pass through A, B, C, respectively, and whose vertices lie on the sides p, q, r.

The point K_2, in the same way, would serve as starting point of a triangle $K_2L_2M_2$ satisfying the given conditions.

If the self-corresponding points K_1 and K_2 should coincide, there is but one solution to the problem, and if they are imaginary there is no real triangle which satisfies the given conditions. On the other hand, the triangles ABC and PQR

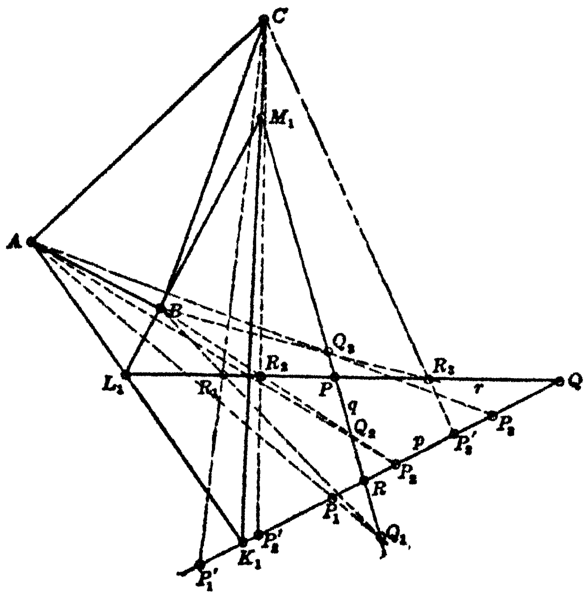


FIG. 132

may be so situated that the two projective ranges of points, p and p', have more than two, and consequently all, of their points self-corresponding. In this case, any point of p may be used as starting point, and there is an unlimited number of triangles inscribed to one and circumscribed to the other of the given triangles.

The problem of this Article may be made more general if stated as follows.

PROBLEM. *To construct a simple n-point whose vertices shall lie in order on n given lines and whose sides shall pass in order through n given points.*

In this case, it is not necessary that the points and lines shall all lie in the same plane; only that each point and the two successive lines shall be co-planar so that the successive projections are possible. The resulting n-point may thus be a skew figure.

222. A Line intersecting Four Given Lines.

PROBLEM. *To find a straight line intersecting four given straight lines, no two of which lie in the same plane.*

If a, b, c, d, are the four given skew lines, the pencils of planes whose axes are a and b may be related perspectively to the range of points c, in which case they will mark out two projective ranges of points on the line d. These ranges of points, in general, have two self-corresponding points, in each of which homologous planes of the pencils a and b cut the line d. The line of intersection of either of these pairs of homologous planes meets all four of the given lines a, b, c, d.

If the self-corresponding points in the projective ranges on the line d are real and distinct, there are two real lines which intersect the four given lines; if they coincide, there is one real line meeting all four, and if imaginary, there is no such line.

Since three of the given lines determine a regulus of the second order, the problem as stated is the same as the following.

PROBLEM. *To find the points in which a given line d meets the ruled surface of the second order determined by three given skew lines a, b, and c.*

If d lies on the ruled surface and belongs to the regulus on the surface to which a, b, and c, belong, there is an infinite number of lines meeting all four, and if d belongs to the other

regulus, it lies in a plane with each of the others and, consequently, itself meets each of the others.

223. Triangle Inscribed in a Conic whose sides pass through Fixed Points.

PROBLEM. *To inscribe a triangle in a given conic, whose sides pass through three given points.*

Let S_1, S_2, S_3 (Fig. 133) be the three points through which the sides of the required triangle shall pass, and in this

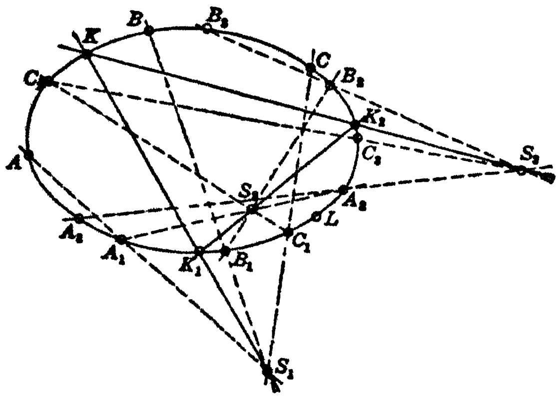


FIG. 133

problem we shall assume that the points are not collinear. On the given conic choose any three points A, B, C, and from S_1 project these chosen points into other points A_1, B_1, C_1, on the conic. From S_2 project A_1, B_1, C_1, into A_2, B_2, C_2, on the conic, and these in turn project from S_3 into A_3, B_3, C_3, on the conic.

The points A, B, C, and A_3, B_3, C_3, determine a projectivity on the conic (§ 154) whose self-corresponding points are, say, K and L. Then S_1K will determine a point K_1 on the conic, which joined to S_2 determines K_2 on the conic, and this in turn joined to S_3 gives a point K_3 on the conic, which

coincides with K. The triangle KK_1K_2 satisfies the required conditions, and LL_1L_2, similarly determined, is a second solution.

If the double points, K and L, of the projectivity on the conic coincide, there is but one triangle which fulfills the given conditions, and if they are imaginary, there is no solution.

224. Triangle Inscribed in a Conic whose sides pass through Three Fixed Collinear Points. In the problem of § 223, if the three given points, S_1, S_2, S_3, lie on one straight line, the solution takes a different form.

Desargues' theorem (§ 176) states that a straight line intersects the sides of a quadrangle inscribed in a conic in points of an involution, of which its intersections with the conic are a conjugate pair. If the quadrangle is reduced to a triangle and a tangent at one vertex, the theorem takes the following form.

THEOREM. *If a triangle is inscribed in a conic, any straight line intersects the conic and two sides of the triangle in pairs of points determining an involution of which the intersections of the line with the third side and the tangent at the opposite vertex are conjugate points.*

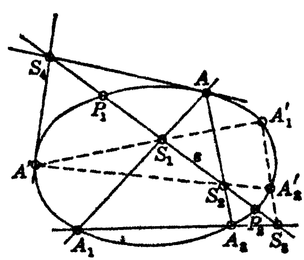


FIG. 134

If the line s (Fig. 134) on which S_1, S_2, and S_3 lie, intersects the given conic at P_1 and P_2, two of the given points, say and S_1 S_2, together with P_1 and P_2 determine an involution on the line in which the point conjugate to S_3 can be determined. Let this conjugate point be S_4 and let tangents from S_4 meet the conic at A and A'. If S_1A intersects the conic a second time at A_1 and S_2A intersects it a second time at A_2, by the theorem just stated the

line A_1A_2 will pass through S_3, the third given point. Hence the triangle AA_1A_2 is the required triangle.

By making use of the point A' in the same way in which A was used to determine the triangle AA_1A_2, a second triangle $A'A'_1A_2'$ may be found fulfilling the required conditions.

In this solution it is assumed that the line s intersects the given conic in real points, and consequently, that in the involution the conjugate to S_3 can be determined by methods already discussed. If the conjugate point S_4 should fall within the conic, no solution is possible. This will happen if all three of the points, S_1, S_2, S_3, lie inside the conic, or if one of them lies inside and the other two outside.

225. Conjugate Elements common to two Superposed Involutions.

PROBLEM. *Two involutions lie on the same base; it is required to find a pair of elements which are conjugate to each other in both involutions.*

The two given involutions may be assumed to lie on the same conic, since every case may be reduced to this by projection or otherwise.

Take any two points A and B on the conic (Fig. 135), conjugate respectively to A_1 and B_1 in the first involution and to A_2 and B_2 in the second involution. Then A and A_1, and B and B_1, are two pairs of conjugate points of the first involution, while A and A_2, B and B_2, are two such pairs in the second involution. The intersection, therefore, of the lines AA_1 and BB_1, or P, is the center of the first involution, and the intersection of the lines AA_2 and BB_2, or Q, is the center of the second involution on the conic.

Since all pairs of points are conjugate which are collinear with the center of an involution on a conic, the line PQ intersects the conic in points which are conjugate in both involutions.

If either center P or Q lies inside the conic; that is, if either involution is elliptic, the line PQ intersects the conic in real points and the two points which are conjugate to each other in both involutions are real.

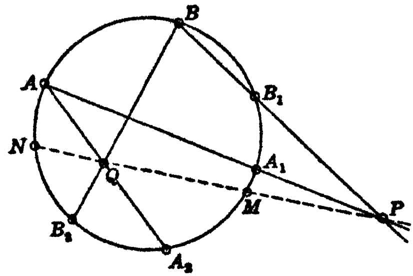


FIG. 135

If, however, both involutions are hyperbolic, the centers lie outside the conic and the line PQ does not necessarily cut the conic. It may cut the conic in two real points; or it may be tangent to the conic, in which case the point of contact is a double point in each involution; or it may lie wholly outside the conic, in which case the doubly conjugate points are imaginary. This latter case arises only when the axes of the two involutions intersect inside the conic; that is, when both involutions are hyperbolic and their double points are so situated on the conic as to separate each other. The following theorem may therefore be stated.

THEOREM. *In any two superposed involutions, there is one pair of elements which are conjugate in both involutions except when the involutions are both hyperbolic and their double elements separate each other.*

If the involutions under consideration are in two superposed pencils of rays of the first order of which one is rectangular and hence elliptic, the doubly conjugate rays are at right angles and we have again the following theorem (see § 170).

THEOREM. *In any involution in a pencil of rays of the first order, there is always one pair of real conjugate rays at right angles.*

The pairs of conjugate diameters of an ellipse or a hyperbola form an involution at the center of the conic (§ 164). To find the axes of an ellipse or a hyperbola, therefore, when two pairs of conjugate diameters are given, it is necessary only to construct the doubly conjugate rays in the pencil of conjugate diameters and a rectangular involution having the same center.

If two superposed involutions have two pairs of conjugate elements in common, the involutions are identical since two pairs are sufficient to determine an involution. From this it follows that if two pairs of conjugate rays of a pencil in involution or two pairs of conjugate points of a range in involution are polar conjugates relative to a conic, all pairs of conjugate elements in these involutions are polar conjugates relative to the conic.

226. Pair of Conjugate Rays of a Pencil in Involution Harmonically Separated by Given Points.

PROBLEM. *In a pencil of rays in involution it is required to determine a pair of conjugate rays which are harmonically separated by two given points of the plane.*

If the given points, M and N, lie in a straight line with the center of the pencil S, there is evidently no solution; and if SM and SN coincide with the double elements of the given involution, every pair of conjugate rays in the involution are harmonically separated by these rays. Moreover, if one of the points M, N, lies on a double element of the involution while the other does not, there is no solution.

In the more general case, however, suppose the point involution on the line MN of which M and N are the double elements is projected from S, there will then be two superposed involution pencils with center S, and we need

only to determine the pair of doubly conjugate rays in the two pencils in order to find the rays of the given involution harmonically separated by M and N. These rays will be real except when the given involution is hyperbolic and its double elements are separated by M and N.

227. The Conic Circumscribing a Triangle for which Pairs of Points of an Involution are Polar Conjugates.

PROBLEM. *Given a triangle and a range of points in involution in the same plane, it is required to construct a conic circumscribing the triangle for which the pairs of conjugate points of the involution are polar conjugates.*

Let ABC be the given triangle (Fig. 136), and let u be the base of the given point involution. It must be assumed

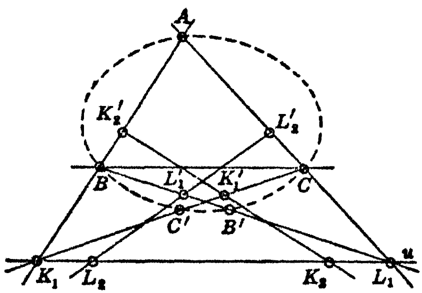


FIG. 136

that u does not pass through any vertex of the triangle and that neither does a double point of the involution lie on a side of the triangle, for if either of these relations should exist, no conic could be constructed satisfying the given conditions.

Suppose u intersects the sides AB and AC of the given triangle at the points K_1 and L_1, and that the conjugates of these points in the given involution are K_2 and L_2, respectively; also, let the harmonic conjugate of K_1 relative to the vertices A and B be K_2', and the harmonic conjugate

of L_1 relative to A and C be L_2'. Then, since K_2 is the polar conjugate of K_1 with respect to the required conic and K_2' is on the polar of K_1 relative to any conic through A and B, K_2K_2' is the polar of K_1 with respect to the required conic, and similarly, L_2L_2' is the polar of L_1.

If the line BL_1 is drawn to intersect the polar of L_1 at L_1', the required conic will pass through the point B', the harmonic conjugate of B relative to L_1 and L_1' (§ 103). Also if CK_1 is drawn to intersect the polar of K_1 at K_1', the required conic will pass through the point C', the harmonic conjugate of C relative to K_1 and K_1'.

For the conic through the five points A, B, C, B', C', there are two pairs of conjugate points, K_1, K_2, and L_1, L_2, of the given involution, which are polar conjugates and, consequently, all pairs of conjugate points of the involution are polar conjugates (§ 225). This conic, therefore, fulfills the conditions of the problem.

It will be observed that the construction here given for the required conic does not depend on whether the double points of the given involution are real or imaginary. If the double points are real, the required conic passes through them since they are self-conjugate, and the same may be said if the double points are imaginary. The problem may then be stated as follows.

PROBLEM. *To construct a conic passing through five given points, three of which are real and the remaining two are the double points, real or imaginary, of a given involution on a straight line.*

228. Conic through Five Given Points of which one, at Least, is Real.

PROBLEM. *To construct a conic through five given points, of which one, at least, is real and the others, in pairs, are the double points, real or imaginary, of two given involutions on intersecting straight lines.*

This problem may be stated equally well in the following form.

PROBLEM. *Through one real point, to construct a conic with respect to which the pairs of conjugate points of two given involutions on intersecting lines are polar conjugates.*

Let P be the given real point and suppose the given involutions lie on the straight lines u and u' intersecting at A (Fig. 137), and that the involutions are determined by two pairs of conjugate points A_1, A_2; B_1, B_2, and A_1', A_2'; B_1', B_2', respectively, A_1 and A_1' coinciding at A.

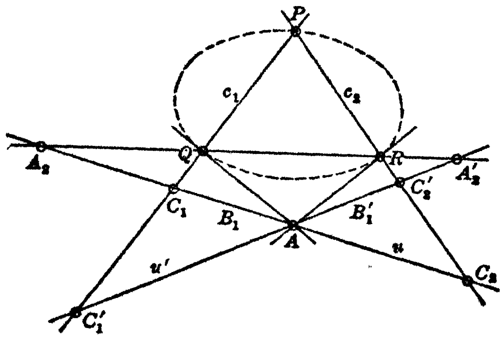


FIG. 137

The given involutions are projected from the point P by two involution pencils of rays in which there will always exist one pair of real rays conjugate to each other in both involutions (§ 225).[1] Suppose these doubly corresponding rays are c_1 and c_2, intersecting the lines u and u' in points

[1] An exception to this statement arises in case both given involutions are hyperbolic with double points M, N, and M', N', respectively, and the rays projecting the pairs of double points from P separate each other. This case can be avoided and the solution here presented be made applicable by constructing two different involutions on straight lines of which M, M', and N, N', are the double points, respectively. In these involutions the pairs of rays projecting the double points from P will not separate each other and the pair of common conjugate rays through P will be real. In this case, moreover, the five given points are real and no special construction is necessary.

C_1 and C_1', C_2 and C_2', respectively. Then C_1 and C_2 are polar conjugates with respect to the required conic, as are also C_1' and C_2'.

Since the point A is conjugate with respect to the required conic, to both A_2 and A_2', the line A_2A_2' is the polar of A. Let this polar intersect the rays c_1 and c_2 at points Q and R, respectively. Then Q and R are points of the required conic to which AQ and AR are tangents.

For since A is the pole of one side of the triangle PQR relative to the required conic, and lines through A intersect the other two sides in conjugate points with respect to that conic, the triangle PQR is inscribed in the conic (§ 115), and AQ and AR are tangents at the two vertices. The conic is, therefore, fully determined, three points of it and the tangents at two of them being known.

229. Conics Circumscribing a Triangle for which Pairs of Conjugate Rays of a Pencil in Involution are Polar Conjugates.

Let ABC be the given triangle and let S be the center of the given involution (Fig. 138). It is evident that the required conic cannot pass through S; also that S cannot coincide with any vertex of the given triangle, for in that case all rays of the given involution would be conjugate to a single ray; namely, the tangent to the conic at that point. There are, therefore, at least two sides of the given triangle, AB and AC, say, on which S does not lie.

Moreover, the double rays of the given involution are self-conjugate with respect to the required conic; that is, they are tangents to the conic, and if the double rays are real and either of them passes through a vertex of the given triangle, the problem reduces to the construction of a conic tangent to a given line and passing through three points at one of which there is a given tangent. In this form the

problem is a special case of that solved in § 220. If both real double rays of the given involution pass through vertices of the triangle, the required conic will have given three points and the tangents at two of them, and it is therefore fully determined.

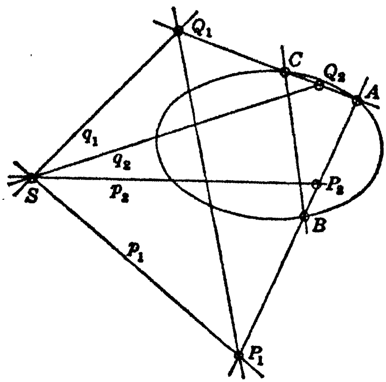


FIG. 138

To solve the more general problem, we determine in the given involution the pair of conjugate rays, p_1 and p_2, which are harmonically separated by the points A and B (§ 226), and let these rays intersect the side AB in the points P_1 and P_2, respectively. Also, in the given involution, determine the pair of conjugate rays, q_1 and q_2, harmonically separated by A and C, and let them intersect the side AC at Q_1 and Q_2. These pairs of conjugate rays are real except, for the first pair, when the given involution is hyperbolic and the double rays separate A and B, and, for the second pair, when the double rays separate A and C. In either of these cases, no conic can be constructed satisfying the given conditions.

For any conic through the points A and B, the polar of P_1 passes through P_2, and *vice versa* (§ 103); also, for any conic through A and C, the polar of Q_1 passes through Q_2. Con-

struct the conic through the vertices of the given triangle ABC, for which the given point S is the pole of the line P_1Q_1 (Chap. IX, Ex. 7, page 134). For this conic, the polar of P_1 passes through both P_2 and S, hence p_1 and p_2 are conjugate rays. Also, for this conic the polar of Q_1 passes through both Q_2 and S, hence q_1 and q_2 are conjugate rays. Since two pairs of homologous rays in the given involution are polar conjugates with respect to this conic, all pairs are polar conjugates (§ 225), and the conic satisfies the given conditions.

The conic through the points A, B, C, for which S is the pole of the line P_1Q_2, or for which S is the pole of the line P_2Q_1, or the line P_2Q_2, would likewise satisfy the given conditions, for in any of these cases p_1 and p_2, q_1 and q_2, are conjugate rays relative to the conic.

There are thus four conics which circumscribe a given triangle for which the pairs of homologous rays in the given involution are polar conjugates. The uniqueness of the construction throughout makes it clear that there are not more than four conics satisfying the given conditions.

Since the double rays of the involution in the given pencil are self-conjugate relative to the required conic and, consequently, are tangents to that conic, the following theorem may be stated.

THEOREM. *Through the vertices of a triangle, four, and only four, conics may be constructed which are tangent to each of two given lines. These conics are real, provided the given lines separate no two vertices of the triangle.*

If the given involution is rectangular, its center S is a focus of the required conic (§ 188), and the following theorem may be stated.

THEOREM. *Through the vertices of a given triangle, four, and only four, conics may be constructed having a given point for focus.*

EXERCISES

1. Given a simple plane pentagon, draw a second pentagon which is both inscribed and circumscribed to the given one.

2. Circumscribe a triangle about a given triangle so that two of its vertices shall lie on two given straight lines and the angle at the third vertex shall be of given magnitude.

3. Through a given point draw two straight lines which will intercept equal segments on two given lines.

4. Choose a point in each of two given straight lines such that the line joining them subtends a right angle at either of two given points.

5. On a given straight line determine a segment which subtends a given angle at either of two given points.

6. Given an involution on a straight line, find in it a pair of conjugate points equidistant from a given point of the line.

7. In two projective ranges of points on the same straight line find a pair of homologous points which are a given distance apart; or, in two concentric pencils of rays projectively related, find a pair of homologous rays which make a given angle with each other.

8. A conic has in general one pair of conjugate diameters parallel to a pair of conjugate diameters in another conic in the same plane. If both conics are hyperbolas and the directions of their asymptotes separate each other, these pairs of parallel conjugate diameters are imaginary.

9. In a plane there are given a triangle and a pencil of rays in involution; construct a conic touching the three sides of the triangle for which the pairs of conjugate rays in the involution are polar conjugates. In other words, construct a conic having given five tangents of which three are real and the other two are either real or conjugate imaginaries.

10. Given a triangle and a range of points in involution in the same plane, it is required to inscribe a conic in the triangle for which the pairs of homologous points of the involution are polar conjugates; or,

In a given triangle, inscribe a conic which shall pass through two given points in a plane, real or imaginary.

11. All circles in a plane intersect the infinitely distant line in the same two conjugate imaginary points.

12. In an elliptic involution on a straight line, there are two pairs of conjugate points which harmonically separate each other.

CHAPTER XVI

THE THEORY OF INVERSION

230. Polar Reciprocation Relative to a Circle. The polar reciprocal of a plane figure relative to a circle in its plane has interesting properties not possessed by its reciprocal relative to any other conic, and for this reason polar reciprocation with respect to a circle may well receive special consideration.

In the plane of a given circle whose center is O and radius r (Fig. 139), the polar of any point P with respect to the

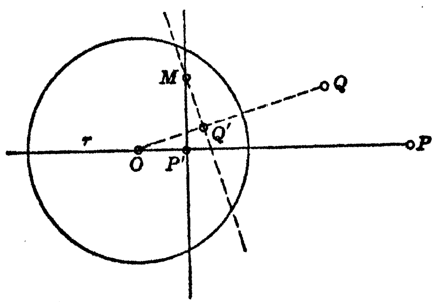


FIG. 139

circle intersects the line OP at right angles at a point P' such that (§§ 103, 37) we have

$$OP \cdot OP' = r^2.$$

For any given circle, therefore, the product $OP \cdot OP'$ is constant, and as the point P takes up different positions in the plane, the vector or radius OP' varies inversely as the vector or radius OP. The point P' is said to be the *inverse* of the point P relative to the given circle, and, similarly, the point P is the inverse of the point P'. The straight line

joining two inverse points in the plane passes through the center of the circle and the product of their distances from the center is constant.

Since the radius OP' equals a constant multiple of the reciprocal of the radius OP, the plotting of pairs of points inverse to each other relative to a given circle is sometimes called the process of ***reciprocation,*** or the process of ***inversion,*** and the line-segments OP and OP' are called ***reciprocal radii.*** The radius r of the given circle is the ***radius of inversion,*** the center O is the ***center of inversion,*** and the given circle is the ***circle of inversion.***

To any point of the plane there is an inverse point relative to a given circle. A point of the circle of inversion is its own inverse; to a point outside the circle, a point inside is inverse, and *vice versa*; the infinitely distant points of the plane are all inverse to the center of inversion. If the point P describes a figure in the plane, the point P' describes the inverse figure.

231. The Inverse of a Straight Line. If P and Q are any two points of the plane, the angle POQ equals the angle $P'MQ'$ where P' and Q' are the inverse points of P and Q, respectively, and M is the point of intersection of the polars of P and Q; that is, M is the pole of the line PQ relative to the circle of inversion.

If, then, the point P moves along a fixed line PQ, or m (Fig. 140), its polar rotates about M the pole of m and describes a pencil of rays such that the angle between any two rays is equal to the angle made by the homologous rays in the pencil described by OP about the center O. Moreover, homologous rays in these two pencils are perpendicular to each other and intersect at the point P', the inverse point of P. The point P', therefore, describes a circle of which OM is a diameter. From this we may state the following theorem.

THEOREM. *The inverse of a given straight line relative to a fixed circle in the same plane is a circle which passes through the center O of the fixed circle and of which OM is a diameter, where M is the pole of the given line relative to the fixed circle.*

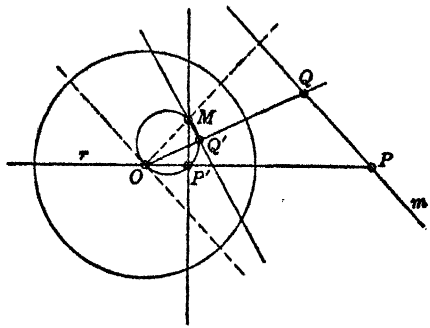


FIG. 140

The inverse of a line tangent to the circle of inversion at any point K is a circle on OK as diameter; the inverse of a line cutting the circle of inversion at two real points is the circle through these points and the center of inversion; the inverse of a line through the center of inversion is the same line, every point of the line being inverse to another point of it. Conversely, the inverse of a given circle through the center of inversion is a straight line.

232. Inverse Figures in a Plane are Isogonal. If a is any given line of the plane and a' is the circle inverse to it, relative to a fixed circle whose center is O, the center of a' lies on the perpendicular from O to the line a, and hence the line through O parallel to a is tangent to the circle a'. If then a and b are any two lines of the plane, the angle between them is equal to the angle made by the circles inverse to them at the center O, and also equal to the angle at their second point of intersection O_1, the point inverse to the intersection of a and b. A triangle whose sides are a, b, c,

is equiangular to the inverse triangle ABC whose sides are arcs of the circles inverse to the lines a, b, c. Indefinitely small triangles in the plane inverse to each other, therefore, are equiangular, and the plane is transformed by inversion into itself in such a manner that corresponding small portions of homologous figures are similar. Such a transformation of the plane is said to be isogonal.

233. Systems of Lines Inverse to Each Other. The inverse of a given figure relative to a circle will vary in size, but not in shape, with a change in the radius of the circle of inversion, and in case magnitude is not an essential factor in the inversion, the radius of inversion is not significant; only the center of inversion needs to be taken into account. Inversion relative to a given point, therefore, is frequently referred to, though in fact the reciprocation is relative to a circle of which the given point is the center.

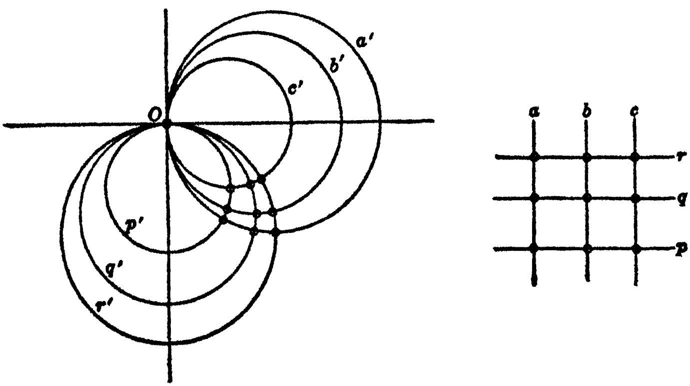


FIG. 141

The inverse of a straight line relative to a point is a circle passing through the point, and the inverse of a system of lines through a fixed point is a system of circles through two points, namely, through the center of inversion and the

inverse of the point common to the given lines. Pairs of circles in the inverse system will intersect at angles equal to those made by the lines to which they are inverse.

A system of parallel lines reciprocates into a system of circles tangent to each other at O, the center of inversion. The centers of these circles lie on the line through O perpendicular to the system of parallel lines. If there are given in the plane two systems of parallel lines at right angles (Fig. 141), they transform by inversion into two systems of circles through the center, the circles of each system being tangent to all the others of that system at the center of inversion and intersecting the circles of the other system at right angles.

234. Peaucellier's Cell. The inversion of small areas of a plane may be effected by a simple mechanical device, a linkage invented in 1873 by Peaucellier, an officer in the French army. This linkage consists of rods as in Fig. 142, freely jointed at the points O, A, B, P, P', such that OA equals OB and AP, PB, BP', $P'A$, are all equal.

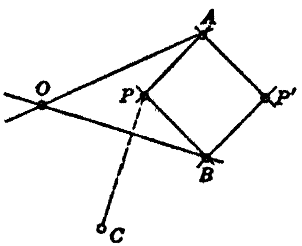


Fig. 142

The points O, P, P', are collinear in whatever position the linkage may be placed, and the product $OP \cdot OP'$ is constant. Hence, if O is the fixed center of inversion, P and P' are inverse points, and as P describes any figure, P' will describe the inverse figure.

Suppose a point C is chosen in the plane of the linkage such that CO equals CP, and the point P moves subject to this restriction, that is, P describes a circle with center C passing through O. Then P' describes a straight line, the inverse of the circle described by P. With this instrument,

therefore, a straight line may be drawn without the use of a ruler, assuming only the fixed length CP and the fixed points C and O.

235. Cross-Ratios are Unaltered by Inversion.

THEOREM. *The cross-ratio of four points on a straight line equals the similar cross-ratio of their inverse points.*

If the given points are on a line through the center of inversion, the points inverse to them lie on the same line and on the polars of the given points relative to the circle of inversion. They are therefore conjugate to the given points and the cross-ratio of the four inverse points equals the similar cross-ratio of the given points (§ 110). The two sets of points form an involution on the line.

If the line on which the given points lie does not pass through the center of inversion, the four inverse points are on the circle inverse to the given line and they are projected from M, the pole of the given line, by the polars of the given points. The cross-ratio of these polars equals the similar cross-ratio of the four points.

In particular, if four points on a straight line through the center of inversion are harmonic, their inverse points on the same line are harmonic, and if the line on which the given points lie does not pass through the center of inversion, the inverse points are projected from any point of the circle inverse to that line, by four harmonic rays.

236. The Inverse of a Circle.

THEOREM. *The inverse of a given circle relative to a fixed circle is again a circle.*

If the center of the fixed circle, that is, the center of inversion, lies on the given circle, the inverse is a straight line, as was seen in § 231. Otherwise, let O be the center of inversion (Fig. 143), and let C be the center of the given circle of which P and Q are any two points collinear with O. Let P' and Q' be inverse to P and Q, respectively.

Then $OP \cdot OP' = r^2$, where r is the radius of the circle of inversion. Also, $OP \cdot OQ = t^2$, a constant for any points P and Q of the given circle, collinear with O. Hence

$$\frac{OP'}{OQ} = \frac{r^2}{t^2} = k,$$

where k is a constant. Through P' draw $P'C'$ parallel to QC, to meet OC at C'. Then the triangles $OP'C'$ and OQC are similar, and

$$\frac{OP'}{OQ} = k = \frac{OC'}{OC} = \frac{C'P'}{CQ}.$$

Since CQ is constant, $C'P'$ is constant. Also, OC is constant, and hence, OC' is constant. The locus of P', there-

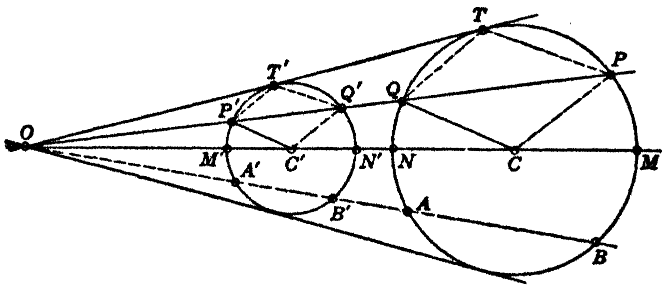


FIG. 143

fore, is a circle of which $C'P'$ is a radius and the fixed point C' is the center. The radius of the inverse circle, $C'P'$, equals k times CQ, the radius of the given circle, and the distance OC' equals k times OC. If the points P and Q coincide at T, so that OT is tangent to the given circle and CQ is perpendicular to OT, since $C'P'$ is parallel to CQ, $C'P'$ will be perpendicular to OT and, consequently, OT is tangent also to the inverse circle at T'. Now

$$OP \cdot OQ = OT^2 = t^2,$$

and

$$OP \cdot OP' = OQ \cdot OQ' = r^2.$$

Hence

$$OP' \cdot OQ' = \frac{r^4}{t^2} = OT'^2.$$

Therefore $OT \cdot OT' = r^2$, and T and T' are inverse points as might have been inferred.

If the line of centers O, C, C' (Fig. 143) intersects the given circle and its inverse at M, N, and M', N', respectively, M and M', N and N' being inverse points, M, C, N, ∞ are harmonic points and to these the harmonic points M', C_1, N', O are inverse, where C_1 lies on the polar of C relative to the circle of inversion (§ 235). The points C_1 and C' do not in general coincide, so that the center of the inverse circle is not, in general, inverse to the center of the given circle.

237. Center of Similitude of Two Circles. In Fig. 143, $C'P'$ was drawn parallel to CQ and, consequently,

$$\frac{OC'}{OC} = \frac{C'P'}{CQ} = \frac{C'Q'}{CP}.$$

Hence, the triangle $OC'Q'$ is similar to the triangle OCP, and $C'Q'$ is parallel to CP.

Similarly $Q'T'$ and $P'T'$ are parallel, respectively, to PT and QT. Moreover,

$$\frac{OQ'}{OP} = \frac{C'Q'}{CP} = \frac{OP'}{OQ},$$

and these relations are true for any line through O intersecting the circles.

The point O, therefore, is a ***centre of similitude*** for the two circles; that is to say, for any line through O intersecting the circles at A, B, and A', B', respectively, we have the relation

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

Hence, the following theorem may be stated.

THEOREM. *If two circles are given whose line of centers is cut externally by their common tangents at a point O, this point is a center of similitude for the circles, and relative to this point each circle is the inverse of the other.*

238. Inversion Relative to an Imaginary Circle. Suppose now we think of any system of points in a plane and their inverse points relative to a circle whose center is O, as lying in two coincident planes, the original points in one plane and their inverse points in the other. Let one of these planes, say the plane in which the inverse points lie, be rotated in its plane about the center of inversion until each point of it has described a semicircle. If P and P' are inverse points in the original position of the two planes, they are collinear with O, and will again be collinear with O after the rotation. But now OP and OP' are measured in opposite senses along the line so that the product $OP \cdot OP'$ is negative. The radius of inversion, therefore, has become imaginary, as has also the circle of inversion, since r^2 equals a negative product.

In such a case, there are no points of the plane which coincide with their inverse points and the inverse of any figure may be derived from the inverse for which the radius of inversion is real, by a reflection through the center O. When the radius of inversion is real, a figure and its inverse lie on the same side of the center of inversion, and when the radius of inversion is imaginary they lie on opposite sides.

In the case of two circles, each of which lies wholly outside the other, there are two pairs of common tangents, one pair intersecting the line of centers externally, and the other pair meeting the line of centers in a point O between them. In this latter case, each of the given circles is the inverse of the other relative to the point O, but the radius of inversion is imaginary. The point is again a center of similitude for the two circles and lines through it are cut proportionally by the circles.

239. A Circle Inverse to Itself.

THEOREM. *If two points of a circle are inverse to each other relative to a fixed point O, all pairs of points of the circle collinear with O are inverse to each other and the circle is inverse to itself.*

In the case under consideration, evidently the center of inversion cannot lie on the given circle. If P and P' are points of the circle inverse to each other, they are collinear with O and $OP \cdot OP' = k$, a constant, positive or negative, according as O lies outside or inside the given circle. Also, any pairs of points of the given circle (Q, Q'), (R, R'), $\cdots$ collinear with O, are inverse to each other since

$$OP \cdot OP' = OQ \cdot OQ' = OR \cdot OR' = \cdots = k,$$

and, consequently, every point of the circle is inverse to another point of the circle. The theorem may be differently stated as follows.

THEOREM. *In the inverse transformation of a plane, any circle through two inverse points is invariant, that is, it is transformed into itself.*

240. Circles which Invert into Concentric Circles.

THEOREM. *Any two circles which do not intersect in real points may be inverted into concentric circles.*

If k_1 and k_2 are the given circles (Fig. 144) intersecting their line of centers at A_1, A_1', and A_2, A_2', respectively, these two pairs of points determine an involution on the line of centers having real double points M_1 and M_2 which harmonically separate both A_1, A_1', and A_2, A_2'. If, then, the circles are inverted with respect to any point on the line of centers, the points inverse to A_1, M_1, A_1', M_2 are harmonic, as are also the points inverse to A_2, M_1, A_2', M_2. If the circles are inverted with respect to the point M_1, the inverse of M_1 is infinitely distant and the inverse of M_2 bisects the segment determined by the inverses of A_1 and A_1'. That is,

the inverse of M_2 is the center of the circle inverse to k_1. Similarly, the inverse of M_2 is the center also of the circle inverse to k_2. Hence these two inverse circles are concentric. Relative to the point M_2, the circles k_1 and k_2 invert into concentric circles of which the inverse of the point M_1 is the center. From this a more general theorem may be stated as follows.

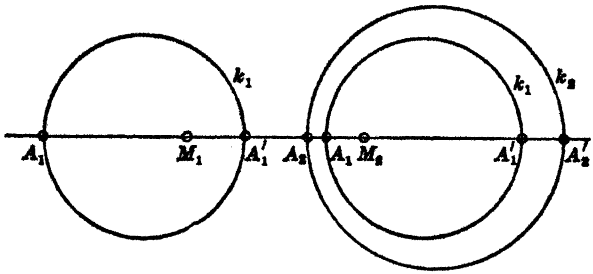


FIG. 144

THEOREM. *If a system of coaxial circles having imaginary intersections is inverted with respect to either of its limiting points, the result is a system of concentric circles having the inverse of the other limiting point as center.*

A system of coaxial circles having real intersections, P and Q, is inverted with respect to one of these intersections, P, into a system of straight lines through a point Q' on the radical axis, inverse to Q with respect to P, the lines of the system being perpendicular to the diameters through P of the several circles.

241. Illustrations of Inversion. As illustrations of how the process of inversion may be used to deduce new geometric properties from known theorems, the following examples will suffice.

1. If P, Q, R, S are any four points of a circle, the angles QPS and QRS are either equal or supplementary.

This is a well-known property of a circle from which a

property of two circles, not so well known, may be derived by inversion. Suppose the circle and the inscribed quadrangle are inverted with respect to the point P (Fig. 145), and the points inverse to Q, R, S, are denoted by Q', R', S', respectively. The given circle inverts into a straight line on which $Q'R'S'$ lie.

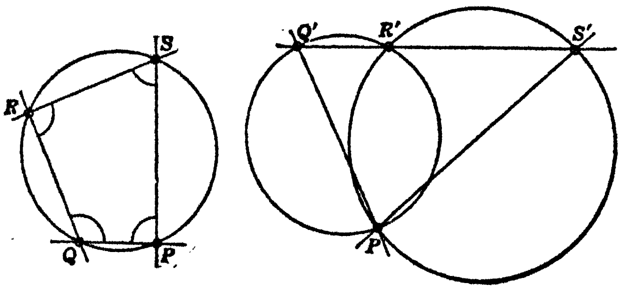


FIG. 145

The line QR inverts into a circle $PQ'R'$.
The line RS inverts into a circle $PR'S'$.
The line PQ inverts into a line PQ'.
The line PS inverts into a line PS'.

The angle made by the lines QR and RS is equal to, or supplementary to, the angle made by the circles $PQ'R'$ and $PR'S'$ at their intersection. The angle made by the lines PQ and PS is equal to the angle made by the lines PQ' and PS'. From this we have the following property of two intersecting circles.

THEOREM. *If two circles intersect at two points P and R', and through one of these points, R', there is drawn any straight line intersecting the circles a second time at Q' and S', respectively, the angle $Q'PS'$ is constant and is equal to, or supplementary to, the angle of intersection of the two circles.*

2. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point. In other words, if ABC

is any triangle and perpendiculars BB_1 and CC_1 from two vertices to the opposite sides intersect at P, then the line AP determined by P and the third vertex is perpendicular to the side BC. Invert the triangle with respect to the point P (Fig. 146), and denote the inverse points by A', B', C', B_1', C_1'.

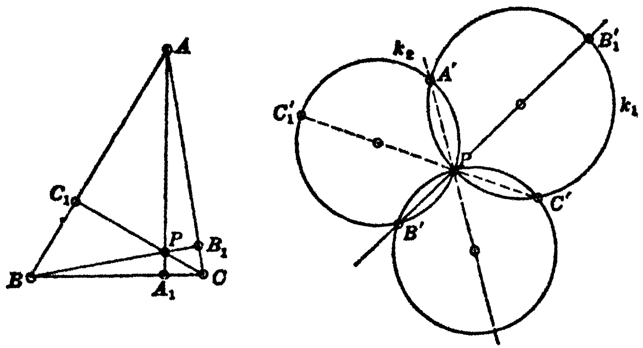


FIG. 146

The line PB inverts into the line PB' and on this line lies B_1'.

The line PC inverts into the line PC' and on this line lies C_1'.

The line PA inverts into the line PA'.

Moreover, the line AB inverts into the circle $PA'B'$ and on this circle C_1' lies. Since PC is perpendicular to AB, PC_1' is a diameter of the circle $PA'B'$. Similarly, the line AC inverts into the circle $PA'C'$ on which B_1' lies and of which PB_1' is a diameter. Also, the line BC inverts into the circle $PB'C'$. Now, since PA is perpendicular to the line BC, PA' produced through P must be a diameter of the circle $PB'C'$. The following theorem may therefore be stated.

THEOREM. *If two circles, k_1 and k_2, intersect at two points P and Q, and through P there is drawn the diameter PR of k_1,*

intersecting k_2 a second time at R'; also, the diameter PS of k_2, intersecting k_1 a second time at S'; then, of the circle $PR'S'$, the line PQ produced is a diameter.

242. Inversion in Space. The points of space may be subjected to an inverse transformation relative to a point, in a manner quite similar to the inverse transformation of a plane. In fact, if a plane in which the points have been correlated by inversion is rotated about a line passing through the center of inversion, and the correlation of inverse points is maintained, every point of space except the center is correlated to an inverse point relative to that center.

In such a correlation, any two points P and P' are inverse if $OP \cdot OP' = r^2$ where O is the center and r the radius of inversion. If r^2 is positive, there is a real locus of points which are inverse to themselves and these lie on a sphere of which O is the center and r the radius. Any point outside this sphere is inverse to a point inside, and all points of the infinitely distant plane are inverse to the center O. If r^2 is negative, there are no points in space inverse to themselves and the inverse of any point P lies on the line OP, but on the opposite side of O.

In the process of inversion of space configurations, any plane through the center inverts into itself, and figures in that plane are inverted into other figures in the plane, as in the preceding articles.

A plane not passing through the center is inverse to a sphere on which the center of inversion lies and to which a plane parallel to the given plane is tangent at the center. In other words, if α is any plane not passing through the center of inversion, and M is the point of α at which the perpendicular from O, the center of inversion, meets the plane, while M' is the inverse point of M, the inverse of the plane α is the sphere α' whose diameter is OM'. For the inverse of a line through M perpendicular to OM is a circle

on OM' as diameter (§ 231), and if this configuration is rotated about OM, the result just stated follows. The plane through O tangent to the sphere α' is parallel to α.

The inverse of any sphere k relative to a center O not lying on it, is a sphere k', and of the two spheres, k and k', the center O is a center of similitude. Any circle in space may be considered as the intersection of two spheres, and its inverse is therefore the intersection of the two inverse spheres, or it is again a circle.

243. Inversion in Space is Isogonal. If ABC is any triangle in the plane α, the points A', B', C', inverse to the vertices of this triangle, will lie on the sphere α' and the inverse of the sides AB, AC, BC, are the arcs $A'B'$, $A'C'$, $B'C'$, marked out on the sphere α' by the planes OAB, OAC, OBC, respectively. The angle made by the lines AB and AC in the plane α is equal to the angle made by the arcs $A'B'$ and $A'C'$ on the sphere α'. For, if $A'B_1$ and $A'C_1$, lying in the planes OAB and OAC, respectively, are tangents to these spherical arcs, they lie in the plane tangent to the sphere at A', and this plane and the plane α are equally inclined to the ray $OA'A$. Similarly, the angles of the triangle at B and C are equal to the angles made by the arcs $B'A'$ and $B'C'$, $C'A'$ and $C'B'$, respectively. Corresponding angles, therefore, in a plane and on its inverse sphere, are equal, and the two surfaces are isogonally related.

Any two planes of space intersect at the same angle as do the spheres inverse to them, for the angle formed by two planes is equal to the angle between their normals drawn from O, the center of inversion, and these are likewise normal to planes through O tangent to the inverse spheres. Any tetrahedron, therefore, is equiangular with the inverse tetrahedron whose faces are the homologous portions of spheres inverse to the faces of the given tetrahedron. Thus, in the process of space inversion, plane angles and dihedral angles

are unaltered and, as a consequence, polyhedral angles are also unaltered except in sense about the vertex.

244. Map Projection. If a sphere σ is inverted relative to a point O lying on it, the result is a plane σ' (§ 242), and all circles on the sphere invert into circles in the plane, except that circles on the sphere passing through O invert into straight lines in the plane. Circles and straight lines in the plane intersect at the same angles as do the circles homologous to them on the sphere.

Systems of circles lying on a sphere and passing through two points, meridians intersecting at the north and south poles, for example, when inverted relative to any point on the sphere, are transformed into a system of circles through two points in a plane, while circles perpendicular to them, parallels of latitude, for example, invert into circles in the plane cutting the former circles at right angles.

In other words, the system of meridians on a sphere when inverted relative to any point of the sphere, form a system of coaxial circles having real intersections in the inverse plane, while the parallels of latitude for that system invert into a system of coaxial circles having the common points of the first system as limiting points of the second. If the center of inversion is at one of the poles, the parallels of latitude invert into a system of concentric circles having the inverse of the other pole as their center, and the meridians invert into straight lines through that point.

By an extension of this process of stereographic projection, geographic maps are constructed in which angles on the map in the minutest parts are equal to corresponding angles on the globe (Mercator's projection), while distances vary according to a changing scale from point to point. In such a projection meridians appear as a system of parallel lines, and parallels of latitude appear as parallel lines perpendicular to them.

EXERCISES

1. Any circle through two inverse points cuts the circle of inversion orthogonally.

2. Show that a circle may be inverted into itself relative to any finite point of its plane not lying on the circle, that two circles may be inverted into themselves realtive to any point of their radical axis, and that three circles may be inverted into themselves relative to one point only.

3. Prove by inversion that the circles having for diameters three chords OA, OB, OC, of a given circle interesct, two and two, in three collinear points.

4. If P' is the inverse of P with respect to a given circle and AB is any chord of the circle through P', prove that PP' bisects the angle APB.

5. Show that any circle, its inverse, and the circle of inversion are coaxial.

6. Three given circles in a plane are in general cut orthogonally by a single circle. Show that if the given circles are inverted with respect to any point on this circle, the centers of the inverse circles are collinear.

7. Given two circles k_1 and k_2 and a straight line intersecting them at points A, B, and C, D, respectively. If O is any point on the radical axis of k_1 and k_2, and OA, OB, intersect the circle k_1 a second time at A', B', and OC, OD intersect the circle k_2 a second time at C', D', respectively, the points O, A', C', B', D', lie on a circle.

8. If two circles cut each of two others orthogonally, the line of centers of either pair is the radical axis of the other pair.

9. If two circles intersect orthogonally, the inverse of the center of the first with respect to the second coincides with the inverse of the center of the second with respect to the first.

10. Two points P and P' are inverse with respect to a circle k. If P, P', and k are inverted with respect to a second circle into P_1, P_1' and k_1, show that P_1 and P_1' are inverse with respect to k_1.

11. If ABC and $A'B'C'$ are inverse triangles relative to a center O, the sum or difference of any two homologous angles, A and A', say, is equal to the angle subtended at O by the opposite sides BC and $B'C'$.

12. If two given circles intersect and cut a third circle orthogonally, their points of intersection are inverse relative to the center of the third circle.

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